**Dynamic investigation of functions using GeoGebra**

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**Abstract.** Functions are a fundamental concept in mathematics education. This article describes the mapping and covariation aspects of functions in one variable and shows dynamic ways of investigating functions with the free educational mathematics software GeoGebra.

**Mapping and Covariation**

Malle (2000) distinguishes between the following two aspects when dealing with functions of the form \( f: x \rightarrow f(x) \).

- **Mapping:** Every \( x \) is mapped to one and only one \( f(x) \).
- **Covariation:** Whenever \( x \) changes, so does \( f(x) \) in a certain way and vice versa.

The local properties of a function are emphasized by the mapping aspect while the covariation aspect makes a more global view necessary. In any kind of representation of functions, both aspects are almost always present. A table of values can be read line by line (mapping aspect) or column by column (covariation aspect). From a graph we can get a specific function value (mapping aspect) and see how \( f(x) \) changes while \( x \) is changing (covariation aspect). Malle also gives some typical examples for questions concerning these two aspects:

- **Mapping:**
  - Which \( f(x) \) belongs to a specific \( x \)?
  - Which \( x \) belongs to a specific \( f(x) \)?

- **Covariation:**
  - How does \( f(x) \) change when \( x \) increases?
  - How do we have to change \( x \) in order to decrease \( f(x) \)?
  - How does \( f(x) \) change when \( x \) is doubled?
  - How do we have to change \( x \) to triple \( f(x) \)?
  - How does \( f(x) \) change when \( x \) is increased by 1?
  - How do we have to change \( x \) to decrease \( f(x) \) by 2?

The free dynamic mathematics software GeoGebra (http://www.geogebra.org) offers several ways to address both of these aspects as we will see below.

**Graph as trace of point \((x, f(x))\)**

Function plotters and computer algebra systems present the whole graph of a function at once. Sometimes, this is an educational disadvantage because it conceals the process of drawing such a graph. One way out of this dilemma is to use spreadsheet software that lets you draw single points from a table of function values. Unfortunately, this way is quite laborious. Most of today’s dynamic geometry systems offer the possibility to use points with coordinates that are calculated using variables. For a given \( x \) we are able to create the point \((x, f(x))\) in such a system. Figure 1 shows how this approach has been realized to visualize the mapping aspect of a function in GeoGebra. Here, the graph evolves as the trace of point \((x, f(x))\) while we move \( x \) with the mouse. So, we don’t get the whole graph at once but see how it is created.
Dynamic table of values

Unlike conventional dynamic geometry software (e.g. *Cabri Geometry*), GeoGebra supports functions of the form $f(x)$ as self-contained objects. Thus, graphs may also be drawn with a single command like with any other function plotting software. Unlike computer algebra systems we are able to use these graphs for dynamic constructions as well. We can create a point on the function’s graph that stays there when dragged with the mouse. While we move the point along the graph, GeoGebra updates its coordinates dynamically. Furthermore, we may also change the $x$-coordinate of such a point directly and see how its $y$-coordinate is adapted to keep it on the function’s graph.

A very convenient characteristic of this approach is its generality: the function $f(x)$ can be subsequently changed. As GeoGebra preserves all relations in the dynamic figure, the point stays on the new function’s graph. Since the coordinates of the point represent one pair of numbers from the function’s table of values, this figure is a *dynamic table of values*: by moving the point along the graph we can dynamically investigate any pair of $x$ and $f(x)$. 

![Figure 1: Graph of a function created using the trace of point $(x, f(x))$ by moving $x$ along the $x$-axis](image1.png)

![Figure 2: Dynamic table values using a point on a function’s graph](image2.png)
Dynagraph

In a *dynagraph* representation (see Goldenberg 1991) the argument $x$ and its function value $f(x)$ are displayed on two parallel lines. This kind of representation emphasizes the covariation aspect as it draws the attention towards the question of what happens with $f(x)$ when $x$ changes. The dynagraph representation of a function can be well realized with dynamic geometry systems very well (see Figure 3): while dragging $x$ with the mouse we may investigate the resulting changes to $f(x)$. This approach is especially useful to examine properties like monotony, linearity, extremal values, domain, co-domain or periodicity.

![Figure 3: Dynagraph](image)

Dynamic Parameter Investigation

Computers can help to discover and explore basic properties of different types of functions. Particularly, they make it very easy to vary parameters and observe the resulting change to the function’s graph. GeoGebra supports such parameter investigations by combining two important features: it offers functions as self-contained objects and provides sliders to change numerical values dynamically using the mouse. A typical scenario of investigating a certain function type with GeoGebra could be realized in three steps:

1. Exploring several examples of the function type using specific parameter values

2. Introducing the parameters using sliders, thereby generalizing the previous examples towards the function type

3. Changing the parameters systematically according to specific questions.

For example, we could investigate the influence of the parameters in a quadratic function to find some kind of connection between these parameters and the roots or the vertex of the parabola.

1. Let’s start with some specific examples like $f(x) = x^2$, $f(x) = x^2 + 2$, $f(x) = x^2 + x$, etc. After entering the first function’s equation into GeoGebra’s input field, the equation is displayed in the algebra window and its corresponding graph is shown in the graphics window. The other examples can be explored by changing the equation in the algebra window.
(2) After getting a first idea about what these parameters might do, we create two sliders \( p \) and \( q \) and redefine \( f(x) \) using the general equation \( f(x) = x^2 + px + q \).

(3) We can now change the parameters \( p \) and \( q \) both in the graphics window using the sliders and in the algebra window by entering a new value directly. Of course, this parameter variation should be done systematically. Some guiding questions on a worksheet can help students to organize their discovery process in this phase.

In figure 4 the trace of vertex S shows the influence of parameter \( p \) on the parabola’s vertex. Interestingly enough, this trace is another parabola again.

![Figure 4: Influence of \( p \) on the vertex of \( f(x) = x^2 + px + q \)](image)

Such dynamic parameter investigations are well-suited for discovery learning (see Bruner 1961) and give the opportunity to have a closer look on special cases. The static representation of a bunch of functions in one image has the disadvantage that it is hard to tell which parameter value belongs to which graph. Whereas in the described approach we always see only one function graph corresponding to the current parameter values of the sliders. All of these steps can be easily done by the students with minimal previous knowledge about the software. This fosters student activity and gives the teacher the opportunity to initiate mathematical experiments.

**Inverse Functions and Relations**

Dynamic mathematics software like GeoGebra offers an interesting way to introduce the concept of inverse functions and relations graphically. Figure 5 shows the inverse relation of \( f(x) = x^2 \) that has been constructed using a locus line. First we place a point \( P \) on the graph of \( f(x) \) and mirror it at line \( g: y = x \) to get \( P' \). Now we create the graph of the inverse relation by letting GeoGebra draw the locus line of \( P' \) with respect to \( P \) moving along the graph of \( f(x) \).

Again, this construction is very general: the equation of \( f(x) \) may be changed afterwards to explore inverse relations of other functions. Furthermore, by dragging \( P \) along the function’s graph we are able to observe the correspondence between the points \( P \) and \( P' \).
New technologies offer new ways of dealing with traditional content in many mathematical areas. Using dynamic geometry software has almost become standard in geometry education. The examples discussed above show that such a dynamic approach can also be used to deal with functions and that dynamic mathematics software like GeoGebra offers several educational advantages compared to traditional methods.

References

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