Influence of dynamic geometry software on plane geometry problem solving strategies

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Agraïments

Abans de tot, moltíssimes gràcies a Josep Maria Fortuny, el meu director de tesi. Agraeixo a Angel Gutiérrez (Universitat de València), Luis Puig (Universitat de València), Matias Camacho (Universitat de la Laguna), José Carrillo (Universitat de Huelva), Philippe Richard (Universitat de Montréal) i Paul Drijvers (Freudenthal Institute) les revisions que m’han ajudat a millorar l’última versió de la tesi.

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1. Introduction

Our research, ‘Influence of dynamic geometry software on plane geometry problem solving strategies’, is framed within the field of Mathematics Education studies. It has been entirely developed within the Programa de Doctorat en Didàctica de la Matemàtica i les Ciències Experimentals of the Universitat Autònoma de Barcelona, Spain, and is part of a broader project on Dynamic Geometry and Education funded by the Ministerio de Ciencia y Educación. We begin by briefly summarizing the problem, the research questions, and the methodology.

It is well-known that computational technologies have a strong impact on the professional practice of mathematics. Nevertheless, their corresponding influence on the teaching and learning of mathematics has remained an ongoing issue. This can be explained by the fact that different obstacles have arisen in the integration of technologies in the classroom. Some of these obstacles are the reticence of certain teachers to include the use of new technologies as well as the lack of teachers training for this integration. Another obstacle which teachers have to confront is how to adjust their pedagogical resources to the possibilities offered by the new resources, in this case software tools. Several questions emerge. What is and what should be the role of technology in the teaching and learning of Geometry? Last but not least, in which way might the use of technology foster the learning of mathematics?

This study focuses on the interpretation of students’ behaviours when solving plane geometry problems, by analysing the relationships among dynamic geometry software use, paper-and-pencil work and the acquisition of geometrical competences. Many pedagogical environments have been created, such as Cinderella (www.cinderella.de), Geometer’s Sketchpad (www.keypress.com/sketchpad), Cabri géomètre II+ (www.cabri.com), and Geogebra (www.geogebra.org), among others. We point to the use of Geogebra, because it is a free Dynamic Geometry Software (henceforth DGS) that also provides basic features of Computer Algebra System to bridge gaps between Geometry, Algebra and Calculus. The software links synthetic geometric constructions (geometric window) to analytic equations, and coordinates representations and graphs (algebraic window). Moreover, according to Hohenwarter and Preiner (2007), GeoGebra appears to be a friendly software that can be operated intuitively and does not require advanced skills to get started.

The integration of technology in the learning and teaching of mathematics requires special attention in many respects. In this research we investigate how the integration of the use of GeoGebra can foster the acquisition of geometric competences in the resolution of problems that compare distances and areas of plane figures. We are interested in studying whether the use of GeoGebra combined with the use of paper-and-pencil (we call such a combination a technological environment) can contribute to the students’ acquisition of geometric competences and the understanding of the concepts of distance and area.

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1 ‘Contribución al análisis y mejora de las competencias matemáticas en la enseñanza secundaria con un nuevo entorno tecnológico’ with reference EDU2008-01963
There are many different difficulties that students face in the learning of these concepts and, more generally speaking, in the resolution of proof problems requiring the coordination of visualization and reasoning processes. Deductive reasoning can be seen as the main difficulty that students face when learning geometry, but they also may have visualization and structural difficulties. They may have difficulties to shift from a geometry based on visual properties to a geometry based on the understanding of properties of objects. It has been argued that dynamic geometric environments tend to promote some types of empirical justifications and inhibit formal justifications, but it provides an environment in which students can freely experiment and it also provides students non traditional ways for the learning of mathematical concepts and methods. De Villiers (1997) states that conviction is necessary to undertake the search for a proof and the software may give insight into the geometric behaviours that can help find a proof. Students can benefit from the integration of dynamic geometry software because it can facilitate visualization, exploring, conjecturing and understanding the distinction between drawing and figure, for instance, and hence can lead to improvements in argumentation abilities.

In order to help students overcome some of these difficulties, the integration of dynamic geometry software in the teaching and learning of geometry should be stressed. As stated by Arzarello, Olivero, Paola and Robutti (2002), task design and teacher orchestration are important issues when encouraging students to go beyond perceptual impression and empirical verification in contexts of dynamic geometry software. Following with this reasoning, we aim at understanding how the integration of GeoGebra, being orchestrated by a teacher, can help to improve geometrical competences by designing adequate tasks to integrate the use of GeoGebra in the classroom.

Laborde, Kynigos, Hollebrands and Strässer (2006) said, “(...) Research on the use of technology in geometry not only offered a window on students’ mathematical conceptions of notions such as angle, quadrilaterals, transformations, but also showed that technology contributes to the construction of other views of these concepts. Research gave evidence of the research and progress in students conceptualization due to geometrical activities (such as construction activities or proof activities) making use of technology with the design of adequate tasks and pedagogical organization. Technology revealed how much the tools shape the mathematical activity and led researchers to revisit the epistemology of geometry” (Laborde et al., 2006, p. 296). We focus on the use of dynamic geometry software in the context of proof problems that compare areas and distances of plane figures, as a way to understand better how the use of GeoGebra can contribute to improving geometric competences.

The two main research questions are:

First research question (Q1):
How can the use of GeoGebra be integrated into a teaching sequence to promote students’ geometric competences (visual, structural, instrumental and deductive)?
Second research question (Q2):
What are the students’ behaviours when solving problems under the influence of the instructional activities, the teacher’s orchestration and the synergy of paper-and-pencil and GeoGebra?

We understand students’ behaviours as the interaction between student-teacher, student-task and student-environment. To approach these questions, we draw on the instrumental approach to tool use (Rabardel, 2001), the cognitive approach (Cobo, 1998), and the concept of hypothetical learning trajectory (Simon, 1995).

We empirically interpret Q1 and Q2 in terms of three goals. In relation to Q1, the first goal points to the development of an instructional design based on problems that compare areas and distances of plane figures to be solved in a technological environment (paper-and-pencil and GeoGebra) and considering teacher orchestration. Still in relation to Q1, the second goal points to the exploration of processes of acquisition of geometric competences in the context of a hypothetical learning trajectory. The learning trajectory is based on a conceptual analysis of the concepts of area and distance and is made tangible in instructional activities that have been tested in a teaching experiment situation. In relation to Q2, the third goal points to the characterization of students’ learning trajectories where the transition processes concerning the acquisition of geometric competences, the teacher’s orchestration and the synergy of the two given environments are fundamental.

The three main research goals are the following:

First research goal (G1):
Designing an instructional sequence of problems based on different itineraries of problems.

Second research goal (G2):
Gaining insight into the acquisition of geometric competences in the context of a hypothetical learning trajectory.

Third research goal (G3):
Analysing and characterising the learning trajectories in terms of the transitions, tutor’s orchestration and synergy of environments

From the perspective of the methodology and the methods, our research consists in a qualitative case study organized around a group of 16-17 year-old students of a high school in Catalonia, Spain. We observed students with a teacher who had been teaching mathematics in this high-school for many years, and who was used to introducing geometry by means of a problem solving dynamics that gave priority to the student's thinking. At the moment of the study, the students were not working on plane geometry or concepts that might directly interfere in the teaching experiment.
We centred on the key concepts of distance and area and focused on problems that compare areas and distances of plane figures, like in the case of Cobo (1998). As we planned to design an instructional sequence based on the resolution of problems, we considered it necessary to work with a type of problem that could be solved with different resolution strategies and a priori were easily adaptable to the students’ needs. We also took into account the role of the dynamic geometry software to adapt and choose the problems. In the instructional design we incorporated pairs of similar problems to be solved only with paper-and-pencil and in a technological environment (paper-and-pencil and GeoGebra), respectively. We wanted to observe and explain the synergy of both environments. One of our assumptions was that the shift between environments fosters the shift from the observation of differences between similar situations to the explanation of similarities.

The students in our study had no experience with dynamic geometry software. To initiate them in the use of GeoGebra, we prepared an introductory session. During the whole teaching experiment, we made them work individually. Each student had his computer and followed similar organizational patterns to those described in Iranzo (2008). We were interested in the instrumental genesis of the software orchestrated by a tutor, which in our case was the teacher in the class. We tried to reduce the effects of the student-student interaction to analyse the student, task, environment and tutor interactions better.

To analyse the influence of the instructional design on the students, it is necessary to identify students’ knowledge related to the contents involved in the problems. Therefore we considered it important to have knowledge of the cognitive characteristics of these students, of the ways in which these kinds of problems have been introduced to them and of the types of answers that were accepted by the teacher.

Among our more relevant findings, we have documented the elaboration of an instructional design and a coding system for the assessment of geometrical competences which have been the source of the characterisation of three prototypical learning trajectories. As the teaching experiment is composed of problems to be solved respectively in a paper-and-pencil environment and in technological environment, the prototypical learning trajectories have been also analysed from the point of view of the synergy of environments.

In this section, we briefly (although not exhaustively) review some of the main lines of research, which are relevant for our study. Before considering the impact of a technological environment in geometry teaching, we should give an account of the mainstream areas of investigation in the process of teaching and learning geometry. Among others, some of them would be Hershkowitz, Parzyasz and Van Dermoelen (1996), and Duval (1998). Duval (1998) identified three kinds of cognitive processes “involved in a geometric activity”: visualisation processes, construction processes by tools and reasoning. These processes appear to be strongly connected and “their synergy is cognitively necessary for proficiency in geometry” (Duval, 1998 p.38).
Concerning geometry in a technological environment, namely in DGS, it is shared among researchers that technology constraints students’ actions and strategies to solve geometric problems. There are several theoretical models to describe students’ interactions with technology. We concentrate our attention on the instrumental perspective developed by psychologists (Vérillon & Rabardel, 1995). The first studies on instrumentation processes dealt with the use of computer algebra system (CAS) by students (Guin & Trouche, 1999). They mainly focused on the difficulties of students using technology. Nowadays instead, as stated by Laborde, Kynigos, Hollebrands and Strässer (2006), “today research pays more attention to the mathematical knowledge involved in instrumental knowledge” (Laborde et al., 2006, p. 280). A very important feature of DGS is the ontological status of mathematical objects in the software. Several researchers have paid attention to notions as dependence of objects (Hoyles 1998, Jones 1996). Objects as points, lines, polygons and the like, during a DGS session become dependent (or partially dependent). Other objects remain free. This ontological distinction of objects turns out to be crucial in the well studied concept of dragging, which is a cornerstone of DGS. The acquisition of the many uses of dragging is according to several researchers a difficult task (Hoyles & Noss, 1994; Sinclair (2003) among others). Sinclair (2003), which observed that students tend to use in a narrow way dragging (namely concentrating on the static figure), demonstrated that usually the dynamic condition was avoided by students. This could be interpreted in terms of instrumentation theory (Vérillon & Rabardel, 1995): the instrumental genesis of the drag mode requires a considerable amount of time. During the instrumental genesis students construct many schemes of utilisation.

The distinction between drawing and figure\(^2\), introduced by Parzysk (1998) and later used by Laborde and Capponi (1994), turns out to be a key concept. According to Strässer (1992), dragging offers mediation between drawing and figure. In a DGS, one works dynamic representations which preserve geometrical properties when dragging. Dragging by its own has been extensively studied by Italian researchers (Arzarello, Olivero, Paola and Robutti (2002); Olivero 2002) in terms of its function in the resolution of conjecture/proof problems. Olivero (2002) characterises several types of dragging (among others): wandering dragging, guided dragging, lieu muet dragging, etc.

Although it is a fact accepted by most researchers that the use of DGS is useful in the processes of visualisation, exploration and conjecturing, there is a controversy about its role on the learning of the necessity of proof. Several researchers such as Jones, Gutiérrez

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\(^2\) Duval (1994) already noticed students’ difficulties to understand the concept of geometric figure and to distinguish it from the drawing: “Mais, en réalité, pour beaucoup d’élèves, les figures ne fonctionnent pas du tout comme cet outil heuristique lors des phases de recherche. La simple vue d’une figure semble exclure le regard mathématique sur cette figure. Deux types de difficultés persistantes sont, en effet, couramment constatés, aux différents niveaux de la scolarité : la résistance à se détacher des formes et des propriétés visuellement reconnues du premier coup d’œil : la figure constitue alors une donnée intuitive qui se suffit à elle-même et qui rend inutile ou absurde, toute exigence de démonstration. (…) ; l’incapacité à voir dans une figure, c’est-à-dire à y discerner des éléments de solution possibles à un problème posé : cela supposerait que l’attention se focalise sur certaines parties de la figure plutôt que sur d’autres, ou que la figure soit éventuellement Guillemhie de tracés supplémentaires. Où il y a tellement d’entrées possibles dans une figure que le choix de l’une d’elle paraît arbitraire, et surtout ce choix semble présupposer que l’on connaisse déjà la solution cherchée ! ” (Duval 1994, pp. 121-122)
and Mariotti (2000), try to assess the validity of the claim that DGS is a helpful environment to the learning of deductive proof. Marrades and Gutiérrez (2000) define following Balacheff (1988) and others, a theoretical framework in which a fine-grained classification of types of proofs is considered. In their work, they are able to document of the quality of justifications. On the other hand, some authors such as Chazan (1993) and Healy (2000) postulate that DGS may become an obstacle in awareness of the necessity of proof due to the "power of conviction" (Rodríguez & Gutiérrez, 2006) of dragging exploration and the facility to provide measurements.

Other very active lines of research in DGS have focused on the importance of the interactions between students, instructors, tasks and technology (Laborde et al., 2006). With respect to tasks and teachers, researchers such as Arzarello, Olivero, Paola and Robutti (2002) argue in favour of the significance of task design and teacher moderation to foster students to understand the need of deductive proof or justification.

At this point, we start our research in DGS (namely with GeoGebra) taking into account the importance of instructional design and the role of teacher guidance. Our study contemplates the careful design of an instructional sequence we call *instructional design* (henceforth ID), which is orchestrated by a teacher (Trouche 2004, Drijvers, Doorman, Boon & Van Gisbergen, 2009). In order to assess the impact on students of the whole learning trajectories of the ID, we explain what we understand by the term *geometrical competence*, and we turn our attention to report changes in students’ geometrical competences in the course of the teaching experiment. We define in fact a fine-grained classification of geometrical competences, which are the visual, structural, instrumental and deductive competence. Given the conditions of the experiment (tutor’s orchestration, the task design ID), a main goal of our research is to understand and characterise the function of DGS in the acquisition of geometrical competences.

The manuscript is structured as follows. The present chapter, ‘Introduction’, contains the research questions and explains the aims of the study. We indicate also with some detail an overview of the state of the art in the field with respect our research. The second chapter, ‘Theoretical framework’ is devoted to the theoretical framework, which addresses four theoretical approaches. These are in order the instrumental approach, the cognitive approach, the concept of hypothetical learning trajectory as a theoretical construct, and finally, the thematic approach which centres the mathematical concepts underlying the research on similarity theory and the theory of euclidean area. In the third chapter, ‘Research design and methodology’, we explain the design of the instructional sequence and the methodological aspects. In the fourth chapter, ‘learning trajectories: profiles, feed-forward and transitions’, we describe and analyse the development of three learning trajectories following the methodology of the previous chapter. In the fifth chapter, ‘interpretation of students’ solutions through the framework’, we interpret the results through the framework. In the chapter, three prototypical learning trajectories are presented and characterised. Finally in the sixth chapter, ‘conclusions and discussion’, we present the results concerning the research questions and discuss the results as well as the limitations of the study and the didactical implications.
2. Theoretical framework
In this section we consider the four main theoretical approaches for our study: the instrumental approach (Rabardel, 2001), the cognitive approach (Cobo, 1998), the hypothetical learning trajectories for designing instructional activities (Simon, 1995) and the thematic approach (Millman & Parker, 1991). Olive, Makar, Hoyos, Kor, Kosheleva, and Strässer (2008) propose a “Didactic Tetrahedron” with four vertices, those of the teacher, the student, the task, and the technology. Moreover, we introduce three complementary aspects: the technological environment, the design of tasks and teacher orchestration.

2.1 The instrumental approach
In this section we comment on the instrumental approach. According to Kieran and Drijvers (2006), the perspective of instrumentation (Trouche 2000, Guin & Trouche 2002) gives a fruitful theoretical framework for understanding the difficulties in the effective use of technology. The instrumental approach to tool use (Rabardel, 2001) has been applied to the study of the integration of computer Algebra System (CAS) into the learning of mathematics as well as to the study of dynamic geometry systems. We apply this framework to the use of GeoGebra, a free dynamic geometry software that also provides basic features of CAS.

Instrumental genesis and schemes
The instrumentation theory partially draws on Vygotsky’s works on tool use (see, for instance, Vygotsky, 1978), as well as those of Engeström (1987), and Rabardel (1995). Rabardel introduces a distinction between the notion of instrument and artefact. According to Rabardel and Verillon (1995), it is important to stress the difference between the artefact and the instrument, which is itself a psychological construct. A machine or a technical system does not immediately constitute a tool for the subject. Laborde, Kynigos, Hollebrands and Strässer say, “it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity” (Laborde et al. 2006, p.279). The process of transforming a tool or artefact into a meaningful instrument is called instrumental genesis. The tool becomes a meaningful instrument after a process of instrumental genesis and during it, while the learner builds up mental schemes (Guin & Trouche 2002; Rabardel 1995; Trouche 2000) as a result of this process. The notion of utilization schemes is close to the Vergnaud-Piaget schemes. Vergnaud (1990) elaborated on the notion of schemes on the basis of the initial notion developed by Piaget: “Appelons ‘scheme’ l’organisation invariante de la conduite pour une classe de situation donnée. C’est dans les schèmes qu’il faut rechercher les connaissances-en-acte du sujet, c’est à dire les éléments cognitifs qui permettent à l’action du sujet d’être opératoire” (Vergnaud 1990, p.136). Drijvers (2003) states that a mental scheme has an intention, a goal, and it contains different components such as “operative invariants”, which stand for the implicit knowledge that is often embedded in a scheme in the form of “concepts-in-action” or “theorems-in-action” (Guin & Trouche, 2002; Trouche, 2000).
The instrumental genesis is a complex process that consists of building up utilization schemes, and depends on the characteristics of the artefact, its constraints and affordances, and on the previous knowledge of the user. Two kinds of utilization schemes can be distinguished (Rabardel, 1995; Trouche, 2000) that refer to two corresponding dimensions of the activity: “les activités relatives aux tâches secondes, c’est à dire relatives à la gestion des caractéristiques et propriétés particulières de l’artefact (…) ; les activités premières principales, orientées vers l’objet de l’activité et pour lesquelles l’artefact est un moyen de réalisation” (Rabardel 1995, p. 91).

We come then to two kinds of utilization schemes: “les schèmes d’usage (Sh. Us.) qui sont relatifs aux tâches ‘secondes’. Ils peuvent, comme dans notre exemple, se situer au niveau de schèmes élémentaires (au sens de non décomposables en unités plus petites susceptibles de répondre à un sous but identifiable), mais ce n’est nullement nécessaire: ils peuvent eux-mêmes être constitués en totalités articulant un ensemble de schèmes élémentaires. Ce qui les caractérise, c’est leur orientation vers les tâches secondes correspondent aux actions et activités spécifiques directement liés à l’artefact; (...) les schèmes d’action instrumentée (Sh. A.I.), qui consistent en totalités dont la signification est donnée par l’acte global ayant pour but d’opérer des transformations sur l’objet de l’activité. Ces schèmes, incorporent, à titre de constituants, les schèmes du premier niveau (Sh. Us.). Ce qui les caractérise, c’est qu’ils sont relatifs aux “tâches premières”. Ils sont constitutifs de ce que Vygotsky appelait les “actes instrumentaux”; pour lesquels il y a recomposition de l’activité dirigée vers le but principal du sujet du fait de l’insertion de l’instrument. Les schèmes du premier niveau (Sh. Us.) constituent, selon la terminologie de Cellérier, des modules spécialises, qui coordonnés les uns aux autres mais aussi avec d’autres schèmes, s’assimilent et s’accommodent réciproquement pour constituer les schèmes d’action instrumentée (Sh. A.I)” (Rabardel, 1995, pp. 91-92).

The first kind of utilization schemes concerns the utility schemes (schèmes d’usage) that come to adapt an artefact for specific purposes, changing or extending its functionality. The second kind comprises the schemes of instrumented actions (schèmes d’action instrumentée), which are defined as coherent and meaningful mental schemes for using the technological tool to solve a specific type of problems. Drijvers (2003) relates the instrumentation to the instrumentalization- the development of schemes of instrumented action and utility schemes- in some cases, and points to the difficulties of establishing a clear distinction. We borrow from him the term instrumented schemes for designing schemes of instrumented action.

In the context of the problems resolution, we have identified instrumented schemes related to the drag tool. We use the terms defined by Restrepo (2008) to name some of these instrumentation schemes. We add some schemes to those found by Restrepo (2008) concerning the tool (instrument) ‘dragging’. For instance, we work with the idea of ‘dragging to find transitivity of equality of ratios by two successive application of Thales theorem’. This is an instrumented scheme that consists of dragging combined with two-steps visual ratio comparison. It requires: a) an awareness of Thales property; b) the mental step of extracting two inside triangles and comparing the common side; and c) an awareness of considering congruent sides. These mental activities give meaning to
technical actions such that: i) select the point P to be dragged along the median; and ii) observe visually or define the measure of the segments on the geometric/algebraic window. We also work with the ideas of: ‘dragging to find the dependence between points’; ‘dragging to verify equidistance’; ‘dragging to find a counterexample”; ‘dragging to obtain equivalent figures’ to apply properties of area function as a measure function; and ‘dragging to verify parallelism’.

As stated by Drijvers (2003), an instrumentation scheme has a technical part that includes the machine actions and mental or cognitive part which gives sense to the technical part. It also requires an understanding of the hidden mathematical properties of the software. In the case of GeoGebra, it requires an awareness of the difference between the GeoGebra and the Euclidean planes. These instrumented schemes have both technical and conceptual dimensions. We use the term technique for the technical part of the instrumented scheme.

**Instrumentation and instrumentalization**

We now characterize the two dimensions of the instrumental genesis: the instrumentation and the instrumentalization. In relation to instrumentation, the affordances and constraints of the software have an influence on the student's problem solving strategies and the corresponding emergent conceptions. Instrumentation involves forming utilization 'schemes' that provide a predictable and repeatable means of integrating artefact and action (Vérillon & Rabardel, 1995). Trouche (2004a) says that instrumentation comprises not only the rules and heuristics for applying an artefact to a task, but also an understanding of the task through which that application becomes meaningful to the user. The instrumentalization process depends on the user and leads to an internalization of the uses of the artefact. The artefact remains the same even if it can be differently instrumentalized.


The artefact becomes an instrument during the instrumental genesis (see Figure 2.1). The user builds up mental schemes, either assimilating already familiar schemes or producing new schemes that allow the user to achieve a cognitive goal. White (2008) says that instrumental genesis makes an artefact meaningful in the context of an activity, and provides a means by which users make give sense to it. A relevant tool of the software is the dragging tool that can also be thought of as an instrument.
Figure 2.1: Instrumental genesis (Figure adapted from Drijvers (2003))

**Dragging tools**

In the analysis of the student's resolution processes we take into account the different uses of the drag tool in the process of problem's resolution. The way dragging is used when solving problems has been primarily studied by Italian researchers who have established a categorization of different types of dragging. In particular, Arzarello, Olivero, Paola, and Robutti (2002), and Olivero (2002) distinguished seven kinds of dragging, in the context of the resolution of an open problem with dynamic geometry software.

- Dragging test, as a way to check whether the geometric construction is correct. For example, after the construction of a rectangle using horizontal and vertical segments, when moving one vertex, it can be checked that the figure becomes a general quadrilateral. This fact can be seen as a criterion of validation for the solution of a construction problem.

- Wandering dragging, after the construction of the figure. The figure is explored by looking for mathematical invariants but without a concrete plan. For example, in the Euler line problem, students drag the vertices of the triangle without any specific plan, trying to observe invariant relationships, in the location of the centre of gravity with respect to the line or the ratio of distances between the three points.

- Guided dragging, when dragging an object in order to get a particular figure. For example in the Varignon theorem students drag the vertices of the initial quadrilateral in order to get a square or a rhombus.

- Dummy locus, when there is a point of the figure that moves along a specific locus. The identification of this locus is generally made with the tool trace. Using locus is particularly fruitful for solving a certain class of geometry problems. For
example, in the problem of inscribing a square in a triangle so that one side of the triangle contains two vertices of the square. By dragging a vertex of the square that lies on a side of the triangle, the fourth vertex of the square moves along a line (applying the tool trace to this vertex). This is an experimental method that shows the construction, which does not suffice as students should justify it.

- **Bound dragging**, if dragging objects that are linked to an object and can only be dragged along this object.
- **Line dragging**, when tracing new points corresponding to the positions occupied by a point of the figure when this figure conserves a certain property.
- **Linked dragging**, when linking a point to an object and dragging this point along it.

Leung and López-Real (2002) stress the key role of dragging when forming a mathematical conjecture and proving by contradiction. We also use the concepts introduced by Olivero (2002) regarding how a student ‘sees’ the result of dragging. He describes the photo-dragging and the cinema-dragging. The photo-dragging consists of “modalities which suggest a discrete sequence of images over time: the subject looks at the initial and final state of the figure, without paying attention to the intermediate instances. The aim is to get a particular figure” (…) [and cinema dragging consists of] “modalities which suggest a film: the subject looks at the variation of the figure while moving and the relationships among the elements of the figure. The aim of dragging is the variation of the figure itself” (Olivero, 2002, p. 141).

The two dragging modalities depend on the problem posed to the student and the student resolution strategy. For instance, the cinema-dragging enables different configurations to be discerned during the exploration phase that may lead to the construction of a proof. This modality, combined with the simultaneous use of the algebraic and geometric windows, allows the students to find algebraic and geometric invariants. Dragging continuously a point linked to an object while focusing on the algebraic window can help distinguish algebraic invariants that are not obvious.

In teaching experiments, the students use the cinema dragging as a way to explore the round-off error behaviour. The students’s goal is to distinguish false negative situations. Chazan (1993) defines this situation as one that presents “empirical data which contradict a true statement” (p. 364).

In relation to the drag tool, we use the term “dragging for adjusting in constructions tasks”. Laborde, Kynigos, Hollebrands and Strässer (2006) say, “constructions done by adjusting are not only part of the solving problem but they scaffold the path to a definite robust construction. They play an important role in moving from a purely visual solution using adjustments to a solution entirely based on theoretical solutions but adjusted by dragging” (Laborde et al., 2006, p.288). They are constructions that generally do not pass the dragging test due to hidden dependencies.

We also use the terms “robust and soft constructions”, as introduced by Healy (2000). This author demonstrated that these two kinds of constructions are complementary. She argues that the general emerges in the exploration of soft constructions and can be
checked by using robust constructions. The cycle from soft to robust and viceversa seems to be a driving force behind students’ generalization processes.

We adapt all the above theoretical terms to proof problems that may require construction tasks as a resolution approach. To do so, we need to comment on another aspect of the integration of dynamic geometry software into a teaching sequence, that of the instrumental orchestration. We assume that the role of the teacher is crucial in the process of instrumental genesis that somehow needs to be guided.

**Instrumental orchestration**

The model for instrumental orchestration was initially developed by Trouche (2004b). Drijvers, Doorman, Boon and Van Gisbergen (2009) have recently modified Trouche’s model considering the necessity of reflecting the actual performance. As stated by Drijvers et al. (2009), an “instrumental orchestration is partially created ‘on the spot’ while teaching” (Drijvers et al., p.147). These authors define an instrumental orchestration as the intentional and systematic organization and use of the artefacts available in the technological environment by the teacher in a given mathematical situation, in order to guide student’s instrumental genesis. They structure the instrumental orchestration into three elements: a didactic configuration, an exploitation mode and a didactical performance.

1. “A didactical configuration is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. These artefacts can be technological tools, but the tasks students work on are important artefacts as well. Task design is seen as part of setting up a didactical configuration.” (Drijvers et al., 2009, p. 146)

In our research, the didactic configuration consists in the instructional design. That is, the selected problems (in terms of the conceptual and procedural concepts involved in the resolution), the decisions about the different tools used for solving each problem (paper-and-pencil, technological environment) and on how to work during the lessons. The students have to solve pairs of similar problems in a technological environment (paper-and-pencil and GeoGebra) and only with paper-and-pencil.

2. “An exploitation mode of a didactical configuration is the way the teacher decides to exploit it for benefit of his didactical intentions. This includes decisions on the way a task is introduced and worked on, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students” (Drijvers et al., p.146).

In our teaching experiment, the students work individually and the teacher may help with messages that have been a priori decided by the researchers. A system of cognitive and contextual messages is designed to help the students during the phases of the resolution process (Schoenfeld, 1985). We adapt the concept of orchestration to both tools: GeoGebra and paper-and-pencil. For each problem we analyse a priori the resolution strategies, the concepts and procedures involved and the instrumented schemes that can
be developed during the resolution in a technological environment. It mainly consists of the decisions on how to exploit it.

In the design of the messages, we analyse the “basic space of the problem” (Cobo, 1998, 2004) to predict students’ actions and difficulties in the resolution process. We then elaborate on a system of messages of three different levels of information. We elaborate on messages for each strategy which respect to the students’ characteristics, and the conceptual and procedural knowledge of each resolution strategy of the basic space of the problem. We define three levels given by messages that: 1) do not content mathematical contents or procedures, 2) only mention the name of the mathematical content or procedures, 3) provide more information about them. We consider specific messages for the familiarization, the verification, and the implementation phases (analysis\exploration, planning\execution), as an adaptation of the set of messages proposed by Cobo (2004). We also consider contextual messages for the problems that have been solved in a technological environment. These messages are created with respect to the usage and instrumented schemes, as well as their epistemic value, and distinguish messages from the different phases. These messages are expected to orient the instrumental genesis of GeoGebra.

First, we have messages that provide information about usage schemes and do not contain mathematical contents or procedures. For instance, we may help the students with the syntax of a given tool, the order of clicks to use the tool, etc. Second, we have messages that refer to the characteristics of the objects and can mention mathematical contents or procedures. For instance, the distinction between free and dependent objects in the use of the dragging tool. Third, we have messages that more deeply inform about a construction and its mathematical contents and procedures, like those about the construction of the exterior height of a triangle, which refer to mathematical contents.

3. “A didactical performance involves the ad hoc decisions taken while teaching on how to actually perform the enacted teaching in the chosen didactic configuration and exploitation mode: what question to raise now, how to do justice to (or to set aside) any particular student input, and how to deal with an unexpected aspect of the mathematical task or the technological tool” (Drijvers et al., p.147).

This last part consists of the decisions taken during the teaching experiment. We work on possible modifications of the messages that have been a priori established when considering some of the students’ difficulties. We take into account unexpected aspects of the tasks and the use itself of GeoGebra. Students may have technical or conceptual difficulties that have not been considered in the second part of the orchestration. In this case, the tutor may help with ad hoc messages to deal with unexpected aspects of the task or the tool. The teacher is asked to follow certain criteria to give concrete messages. Hence the didactical performance involves the ad hoc decisions taken while teaching.

We apply the three-fold model of instrumental orchestration to our research. It consists of an instrumental orchestration of short cycles of individual work in a technological
environment assisted by a tutor (teacher). To characterize this three-fold model Drijvers and his colleagues (2009) say that it is needed to specify the tutor’s behaviour in the context of the teaching experiment.

### 2.2 The cognitive approach

In this section we specify how we use the term knowledge and how we measure the ability of the students to solve the proposed problems in terms of geometric competences. We distinguish between conceptual knowledge and procedural knowledge in the context of this research. We also describe the meaning we give to some terms used in this research such as the terms problem, basic space of a problem, pedagogical space of a problem, logical structure of a problem. These terms are used to explain the conception of the instructional design.

Finally, as we plan to analyse the students’ acquisition of geometric competences we indicate which definition we adopt for the different competences (visual, structural, instrumental and deductive) considered in this research. Finally, we define the complexity level of a problem in the context of problems that compare areas and distances of plane figures (section 2.4 The thematic approach).

**Procedural knowledge and conceptual knowledge**

We distinguish conceptual knowledge as “knowledge that is rich in connections” (Hiebert & Lefèvre, 1986, p.3) from knowledge about isolated facts or pieces of information in the sense described by Cobo (1998). These pieces of information may appear a priori not connected. They become part of the conceptual knowledge insofar as the student is able to establish connections between them. For instance, the area formula of a triangle (base times height divided by two) can be considered as knowledge about facts, or can be considered part of the conceptual knowledge in the case that this information is well connected. The student may connect the area formula of the triangle with the area of other figures (parallelogram, rhombus, etc.) and other forms of the area of a triangle. In this case, we may consider this knowledge as conceptual knowledge. As a matter of fact, Hiebert and Lefèvre (1986) distinguished two kinds of connections: connections of primary level which are constructed at the same level and connections of reflective level.

Pythagoras theorem constitutes an example of reflective connection. The students may connect the relation between Pythagoras theorem and Euclidean area, identifying the area of the squares on the three sides of the right angled triangle. They may also connect Euclidean area and similarity theory (section 2.4 The thematic approach).

Following Polya (1990), we can go further considering generalization and analogy. An example of the former is the generalization of Pythagoras theorem, based on Euclid’s proof of Pythagoras theorem (Euclid’s elements, proposition 31 of book IV): “If three similar polygons are described on three sides of a right triangle, the one described on the hypotenuse the one describe on the hypotenuse is equal in area to the sum of the two others. “The area of the square described on the hypotenuse in I [Figure2.2] is \(a^2\). The area of the irregular polygon described on the hypotenuse in III [Figure 2.3] can be put equal to \(\lambda a^2\); the factor \(\lambda\) is determined as the ratio of two given areas. Yet, then, it follows from the similarity [dilatation] of the three polygons described on the sides \(a\), \(b\) and \(c\) of the triangle in III that their areas are equal to \(\lambda b^2\) and \(\lambda c^2\) respectively. Now, if the equation (A) \([a^2 = b^2 + c^2]\) should be true then also the following equation (B) \([\lambda a^2=\)}
\( \lambda b^2 + \lambda c^2 \) represents a generalization of the original theorem of Pythagoras” (Polya, 1990, p. 17)

![Figure 2.2: Pythagoras theorem (I)](image1)

![Figure 2.3: Generalization (III)](image2)

Another example of connected knowledge among areas and Pythagoras theorem is shown with the following anonym visual proof (Nelsen, 1993) of Pythagoras theorem. We notice the strong similarity between this visual proof and the root problem (Figure 2.4).

![Figure 2.4: Pythagoras theorem visual proof (Nelsen, 1993, p.3)](image3)

Concerning analogy, we consider it in the sense defined by Polya (1990): “Analogy is a sort of similarity. It is, we could say, similarity on a more definite and more conceptual level. Yet, we can express ourselves a little more accurately. The essential difference between analogy and other kinds of similarity lies, it seems to me, in the intentions of the thinker. Similar objects agree with each other in some aspects. If you intend to reduce the aspect in which they agree to definite concepts, you regard those similar objects as analogous. If you succeed in getting down to clear concepts you have clarified the analogy. (...) Two systems are analogous if they agree in clearly definable relations of their respective parts” (Polya, 1990, p.14).
Nevertheless, as stated by Hiebert and Lefevre (1986), in order to establish connections, knowledge of facts is necessary. In the third chapter (Research design and methodology), we define the mathematical contents (conceptual and procedural knowledge) related to the problems considered in the instructional design (ID). We specify in the following paragraph what we understand as procedural knowledge.

Hiebert and Lefevre (1986) define procedural knowledge as knowledge that consists of rules and procedures that are used to complete tasks. They consider that procedural knowledge has two components: the formal language or symbolic representation system and the algorithms and norms to solve the mathematical tasks. We borrow from Hiebert and Lefevre (1986) the notion related to systems of symbolic representations but we extend these terms in the sense defined by Cobo (1998). We distinguish the following components: systems of symbolic representation, technique procedures and algorithms, and heuristic procedures.

Concerning the systems of symbolic representation, we distinguish three key points (Cobo, 1998, p.44).

- The objects to which this kind of knowledge applies, which can be grouped in three classes: on the one hand, standard written symbols (say 4, /, +, etc.), on the other side, concrete objects whose representation may be of algebraic nature- we are concerned here with objects that occur in problems which have been formulated in an algebraic language-; and finally, objects such as mental images or figures.
- Rules governing the use of formal language in mathematics, which can be a source of pedagogical conflict for students with their mathematical intuitions.
- The interpretation of symbols turns out to rely on conceptual knowledge.

Concerning the technique procedures and algorithms, we consider this knowledge in the sense defined by Baroody and Ginsburg (1986), as knowledge about ‘facts’. For instance, in the context of the proposed problems, we identify the following procedures or techniques (composed procedures): identifying and representing the heights of a triangle, applying formulas, applying congruence criteria of triangles, etc.

Concerning the heuristic procedures, the definition given by Puig (1996) of heuristics delimits the heuristic processes as: the study of the behaviours in the resolution process and the means used in the resolution process which are independent of the content and do not necessarily lead to the solution of the problem. The author does not consider in this definition the processes, in which it is necessary to make reference to the content of the proposed problem. In this study we consider the heuristic processes as different ways of reaching a solution of the problem (we use the term resolution strategies). As mentioned above, the strategies specify several paths for solving the problem. The way of solving a problem depends on the kind of problem, the knowledge and the experience of the students. So the resolution of these problems has to consider the search of transformation processes in which specific knowledge and heuristic strategies are intertwined, with the objective of solving the problem.
So far, we have focused on the procedural knowledge and conceptual knowledge in the context of problems that compare areas and distances of plane figures. As we aim at investigating how to integrate GeoGebra in a teaching sequence to promote students geometric competences (visual, structural, instrumental and deductive), we specify in the following section the meaning that we give to these terms.

**Geometric competences**

*Mathematical competency* (OECD, 2006) consists of fundamental mathematical knowledge (facts and skills), mathematical concepts and mathematical abilities (mathematical modelling, reasoning, giving evidence to support mathematical theses or representing mathematical objects). We use the term competence in the Geometry area of solving problems that compare areas and distances of plane figures. Whether or not students have acquired mathematical competency becomes evident when looking at their capacity to use mathematics in that context, and also when looking at their ability to solve the proposed problems. We define four geometric competences, in the context of our research. These geometric competences can be related with some of the competences concerned with the PISA assessment. “An individual who is to engage successfully in mathematisation within a variety of situations extra and intra-mathematical contexts, and overarching ideas, needs to possess a number of mathematical competences which, taken together, can be seen as constituting comprehensive mathematical competence. Each of these competences can be possessed at different levels of mastery. To identify and examine these competences, PISA decided to make use of eight characteristic competences that rely, in their present form on the work of Niss (1999) and his Danish colleagues” (OECD, 2006, p.100).

We consider in this research the visual, structural, instrumental and deductive competences. In the following paragraphs we specify these terms and we relate these four competences with the eight PISA competences. “There is a considerable overlap among these competences and when using mathematics, it is usually necessary to draw simultaneously on many of the competencies” (OCDE, 2006, p.107).

We now go on to specify the meaning that we give to the four geometrical competences (visual, structural, instrumental and deductive). As these terms have been used in various ways in the research literature, we clarify how we use them in this research. In the third chapter (research design and methodology), we define indicators to measure the abilities concerning each competence.

**Visualization competence**

We specify in which sense we use the term visualization competence. We use the term visualization with the meaning given by Gutiérrez (1996): “*I therefore consider ‘visualization’ in mathematics as the kind of reasoning activity based on the use of visual or spatial elements*” (Gutiérrez, 1996, p.9). He considers mental images as the basic element in visualization. Presmeg (1986a, 1986b) identified several mental images, appearing in problem solving, such as: concrete image (“picture in the mind”), pattern imagery, memory images of formulas, kinaesthetic imagery (of physical movement) and dynamic imagery (the image itself is moved or transformed). As stated by Gutiérrez (1996), “*the types of mental images identified by Presmeg (1986) can probably be
completed if research is made to identify them in other specific areas of mathematics like probability, functional analysis, or analytical geometry. Usually only a few types of mental images are necessary to solve a certain kind of task” (Gutiérrez, 1996, p.9).

**Structural competence**
Following Niss’s (1999) definition of the structural competence, we understand this competence as the ability to elaborate the problem. It concerns the ability to give mathematical structure to the problem (modelling). In the PISA assessment the modelling competence is defined as follows: “it involves structuring the field or situation to be modelled; translating reality into mathematical structures in context that may be complex or largely different from what students are familiar with; interpreting back and forth between models (and their results) and reality, including aspects of communication about model results: gathering information and data, monitoring the modelling process and validating the resulting model. It also includes reflecting through analysing, offering a critique and engaging in more complex communication about models and modelling” (OCDE, 2006, p.107). The structural competence also includes immediate conjectures which have the objective of understanding the problem (conjecturing) and definitions (naming and assigning characterizations).

**Instrumental competence**
We understand this competence as the process by which the student is immersed in the task of finding ways of resolution. The student is involved in a task of searching (he faces a problem of proof search). “It involves the awareness and understanding about proof. We understand that a student gets insight in the instrumental competence if “he begins to realize the limitations of empirical arguments as methods for validating mathematical generalizations and see an intellectual need to learn about secure methods for validation (i.e., proofs)” (Stylianides & Stylianides, 2009, p.348).
Some characteristic elements of this process are the following:
- The students look for intermediate results or key-lemmas which are to be understood as a planned strategy of the way of solving the problem.
- The students not only search fundamental properties but also look for possible ways of solving the problem. For instance, reconfiguring figures in the GeoGebra plane through dragging may help the students to sketch a way of proof based on particularization. The students expect to find invariants by exploring through dragging.
- Other examples are the search for counterexamples, supposing the problem solved, etc.

**Deductive competence**
In this study the deductive competence is understood in a rather narrow sense. We consider the ability of the students to produce verbal discourse (formal or informal) but deductively valid.
We differentiate the main categories defined by Balacheff (1998): empirical and deductive justifications depending in whether the justification consists of checking examples or not. Empirical and deductive justifications are split in different classes. We
use Balacheff’s classification introducing some subcategories defined by Marrades and Gutiérrez (2000).

Within empirical justifications there are three classes:
- **Naive empiricism**: the conjecture is justified by showing that it is true in one or several examples, usually selected without a specific criterion. Marrades and Gutiérrez (2000) introduce two subcategories: “perceptual type, when the checking may involve visual or tactile perception and inductive type, when it also involves element or relationships found in examples” (Marrades & Gutiérrez, 2000, p. 91).

- **Crucial experiment**: the conjecture is justified by showing that it is true in a specific, carefully selected example. “We distinguish several types of justifications by crucial experiment, depending on how the crucial experiment is used:
  - Example-based, when the justification shows only the existence of an example or the lack of counter-examples; constructive, in which the justification focuses on the way of getting the example; analytical in which the justification is based on properties empirically observed in the example or auxiliary elements; and intellectual, when the justification is based on empirical observation of the example, but the justification mainly uses accepted properties or relationships among elements of the example” (Marrades & Gutiérrez, 2000, p.92).

- **Generic example**: the justification is based on a specific example, seen as a characteristic representative of its class. “The four types of justifications (example-based, constructive, analytical and intellectual) defined for the crucial experiment are found here too, in descriptions of how the generic example is used in the justification” (Marrades & Gutiérrez, 2000, p.93)

Within pragmatic justifications there are two classes
- **Thought experiment**: a specific example is used to help organize the justification. “Following Harel and Sowder (1996), we can find two types of thought experiments, depending on the style of the justification: transformative justifications are based on mental operations producing a transformation of the initial problem into another equivalent one. The role of examples is to help to foresee which transformations are convenient. Transformations may be based on spatial mental images, symbolic manipulations or construction of objects. Structural justifications are sequences of logical deductions derived from the data of the problem and axioms, definitions or accepted theorems. The role of examples is to help organize the steps in deductions” (Marrades & Gutiérrez, 2000, p.93).

- **Formal deduction**: the justification on mental operations without the help of examples. Marrades and Gutiérrez (2000) also distinguish here two kinds of justifications (transformative and structural).

The majority of students participating in this research produce empirical justifications. There are signs of a transition from empirical to abstract in certain students. We do not expect these students to produce deductive justifications. We take into account this fact to define the coding system concerning the deductive competence.

Finally, we define in the following paragraphs some terms that we will use in the research, such as: problem, basic space of a problem, logical structure of a problem and complexity level of a problem.
Problems and exercises
We consider the term used in a school context. As Mayer (1986) says, any definition of problem should include three ideas: 1) currently, the problem is in a concrete state, but 2) we wish to change the current state of the problem, and 3) there is no direct and obvious path to realise this change of state. According to Cobo (1998), we also should consider the following characteristics: a) the task and the subject are intertwined and b) the task remains the same and the fact of being a problem depends on the subject. The context in which we present the problem should also be considered (for example, a problem may be obvious for the students if they have worked with similar problems before). We should not consider problems in which the connection between the initial state and the final state are based only on procedural aspects such as algebraic calculations. We borrow from Cobo (1998) the different terms considered to discern different kind of problems, such as: “Search exercises”- the student has to remember a result that immediately leads to the resolution; algorithmic exercises-if the student only uses one algorithm automatically-; “application problem” – the student knows the procedure, but he has to justify that it is appropriate, or the execution phase requires an argumentation-; “search problem” – the student has to create the procedures that lead him to reach the solution --; and “problematic situations”- when the objectives are not clear.

The problems considered in the teaching experiment are problems that compare areas and distances of plane figures that are “search problems”. In the following paragraph we define how we have characterised these problems and some terms used in the research to create an instructional design (ID). In section 2.4 Themathic approach we define the main theoretical points of these problems. The problems considered are similar problems, understanding similarity in terms of basic space of the problem and logical structure of the problem. We define these terms in the following paragraphs. To carry out the a priori analysis of the problems we consider the different resolution approaches to solve these problems.

Space of a problem and basic space of a problem
Furthermore, as Cobo (1998) states in reference to Mayer (1986), it is important to consider the difference between the basic space of a problem, in reference to the person that solves the problem and the basic space of a problem as the different paths for solving the problem obtained by a person that solves the problem perfectly. These two basic spaces of the problem may be different. For example, the student's basic space of a problem may be different from an expert's basic space of the same problem. We adapt these definitions to our research. We consider the basic space of a problem as the different paths for solving the problem, which depends on the knowledge of the person that solves the problem and also on the different approaches considered. We use the term basic space of a problem for the basic space of the problem obtained by an expert. And we use the term student's basic space of a problem for the basic space obtained considering all the students resolutions. We use the terms student's basic space of a problem for the paper-and-pencil and the GeoGebra resolution. For the GeoGebra resolutions we also distinguish the necessary tools and techniques to solve the problem with the different resolution strategies of the problem’s basic space.
For designing the basic space of a problem we identify the concepts and procedures involved in the different resolution approaches. “Researchers in the field of mathematical education have attempted to distinguish among knowledge of mathematical concepts, knowledge of mathematical procedures and acquisition of skills (e.g. Baroody, Feil and Johnson, 2007; Hiebert and Lefebre, 1986; Star, 2005, 2007; Gray and Tall, 1994; Tall, Gray, Ali et al., 2001)” (Olive, Makar, Hoyos, Kor, Kosheleva, & Strässer, 2008, p.3). We consider conceptual knowledge as knowledge that involves relations or connections (Hiebert & Lefevre, 1986). We define in the fourth section (2.4 thematic approach) the concepts and procedures involved in the problems that compare areas and distances of plane figures.

**Logical structure of a problem and ‘complexity levels’ of problems**

We also define for each problem its logical structure and three levels of complexity for the problems. These levels are defined considering the complexity of the logical structure, the basic space of the problem (different resolution approaches and key steps) and the difficulty to apply approaches based on equicomplementary dissection rules (congruent figures, similar figures and equivalent figures). These levels make sense only in the context of the set of problems considered which are similar in terms of basic space and logical structure. In the following table (Table 2.5) we represent the figures associated to the statement of the some of the problems. In order to grasp the self-containedness of the problems of instructional design, we look at some striking similarities between almost all the problems. These similarities are obtained by simple continuous deformation of figures that preserve area, by applying reflections. Also we obtain new problems by particularization and generalization.
To define the three complexity levels of the problems we adapt the idea of reproduction clusters defined in PISA. “PISA has chosen to describe the cognitive activities that these competences encompass according to three competency clusters: the reproduction cluster, the connections cluster, and the reflection cluster” (OCDE, p. 112).

<table>
<thead>
<tr>
<th>Problems of level one</th>
<th>Problems of level two</th>
<th>Problems of level three</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard representations and definitions</td>
<td>Non-routine problem solving</td>
<td>Non-routine problem solving</td>
</tr>
<tr>
<td>Routine procedures</td>
<td>Logical structure based on one quantifier</td>
<td>Logical structure based on two quantifiers</td>
</tr>
<tr>
<td>Routine problem solving</td>
<td>Resolution: geometrical approach based on discerning congruent figures, applying equicomplementary dissection rules</td>
<td>Resolution: geometrical approach based on discerning similar, equivalent figures applying equicomplementary dissection rules</td>
</tr>
<tr>
<td>Simple logical structure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>One-step resolution strategies</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Table 2.6: complexity level of the problems*
Pedagogic space of a problem

As we plan to consider the intervention of a tutor to help the students during the resolution process, for each problem we define cognitive and contextual messages (to guide the instrumental genesis of GeoGebra) of different levels. Nevertheless, we plan to help the students by trying to give little information to them. For this reason we classify the messages in three levels. These levels are defined in detail in the third chapter (Design research and methodology). For designing these messages we adapt Cobo’s (2004) system of messages. We consider the basic space of each problem and we try to hypothesize about the student’s conceptual and procedural difficulties for each resolution approach. For instance, for the quadrilateral problem, we show some messages designed a priori. We identify different typologies of messages: conceptual, heuristic, metacognitive and semiotic. Each kind of message can be identified with concrete phases of the resolution process. For instance, metacognitive messages are usually associated to verification phases. We show in the following tables the statement of the quadrilateral problem which has to be solved with paper-and-pencil and some cognitive messages associated to this problem.

Let ABC a triangle and let P any point of the side BC, N and M be the midpoints of the sides AB and AC respectively.

What relation is there between the area of the quadrilateral ANPM and the addition of the areas of the triangles BNP and PMC?

<table>
<thead>
<tr>
<th>Cognitive messages</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptual</td>
<td>Try to find relations between the segments obtained</td>
<td>Could you find the relation between the segments NM and BC?</td>
<td>How could you justify that 2MN = BC?</td>
</tr>
<tr>
<td>Heuristic</td>
<td>Try to split the figure into other figures</td>
<td>Try to split the triangle ABC into triangles</td>
<td>Try to split the triangle ABC into triangle drawing parallel lines or joining vertices</td>
</tr>
<tr>
<td>Metacognitive</td>
<td>Try to revise the solution obtained</td>
<td>Revise the last steps</td>
<td>Revise the expression obtained</td>
</tr>
<tr>
<td>Semiotic</td>
<td>Do you think that you need concrete data to solve the problem?</td>
<td>Try to represent the segments of the figure</td>
<td>Try to express the relation between segments using symbolic representation</td>
</tr>
</tbody>
</table>

Table 2.7: Quadrilateral problem

Table 2.8: Example of cognitive messages for the resolution of the quadrilateral problem

2.3 Hypothetical learning trajectories for designing instructional activities

We decided to consider the concept of hypothetical learning trajectory (Simon, 1995) as it offers a description of key aspects for planning mathematical lessons. As we plan to
produce an instructional design based on the resolution of problems that compare areas and distances of plane figures in a technological environment, the problems have to be adapted to the environment (GeoGebra and paper-and-pencil), to the specific needs of the students and to the characteristics of the software. As stated by Laborde, Kynigos, Hollebrands and Strässer, “some researchers also stress that the choice of the tasks in relation to the affordances of the technological geometry environment may be critical for the development of students ‘understanding. A relevant combination of tools made available to students and of problems situations is generally considered as a good ‘milieu’ (in the sense of theory of didactic situations) for the emergence of new knowledge (Laborde et al., 2006, p.292)”.

Simon (1995) offered the hypothetical learning trajectory as a way to explain important aspects of pedagogical thinking involved in teaching mathematics for understanding. He described how mathematics teachers, oriented by a constructivist perspective and mathematical goals for students, can think about the design and use of mathematical tasks to foster mathematical conceptual learning. They provide a way of producing successful lessons and modifying unsuccessful lessons to promote mathematical conceptual learning. We also borrow the mechanism articulated by Simon and Tzur (2004)-reflection on activity-effect relationships- for explaining mathematics concept development. They summarize the mechanism as follows: “Having set a goal, the learners call on available activity (or set of activities) in an effort to meet the goal. Working toward this goal, the learners attend to the effects of their goal-directed activity. While attending to the effects of their activity (relative to their goal), the learners create mental records. The unit of experience that is recorded is an iteration of the activity linked to its effect. The learners sort and compare, which leads to identifying patterns, that is, relationships among the activity and effects. This reflective abstraction of a new (to the learner) activity-effect relationship is the mechanism by which a new concept is constructed. Implied in this elaboration of reflective abstraction is that each of the components, creating records of experience, sorting and comparing records, and identifying patterns in those records, is an inborn mental ability and tendency (von Glasersferd, 1995)” (Simon & Tzur, 2004, p.94).

For our research, we consider the development of lessons involving the resolution of non-routine proof problems that compare areas and distances of plane figures. We plan to introduce the use of GeoGebra with the expectation that the synergy of environments (paper-and-pencil and GeoGebra) can help the students to improve their geometrical competences in the context of the problems proposed. For instance, Simon (1995) focused on problems to strengthen students’ understanding of area- under what type of changes in shape does the area of a region remain invariant?

We borrowed from Drijvers (2003) the idea of adapting hypothetical learning trajectories (Simon, 1995) to our particular research. Simon (1995) describes how teaching takes place according to a mathematics teaching cycle that includes stating the goals of the teaching. The development of a hypothetical learning trajectory involves the assessment of the starting level of understanding, the end goal and the development of a chain of students’ activities that foster the movement to the goal. These students’ activities are
designed to foster mental activities in the students. The designer gives the description of why the instructional activity is supposed to work and what mathematical development is expected to foster. After a *field test by a teaching experiment* the hypothetical learning trajectory is usually adapted and changed. These changes, which are based on teaching experiences, allow the designer to start a new cycle.

Simon (1995) used the hypothetical learning trajectories for designing and planning short cycles of one or two lessons, and Drijvers (2003) developed hypothetical learning trajectories for experiments that lasted 15 to 20 lessons, taking a researcher’s perspective. In our research, we also take the researcher’s perspective and develop a teaching experiment that lasts 4-6 lessons based on problem solving. As stated by Simon (1995), “the consideration of the learning goal, the learning activities, and the thinking and learning in which students may engage make up the hypothetical learning trajectory” (Simon, 2005, p.133).

The goal of our instructional activities is to improve students’ geometric competences (visual, structural, instrumental and deductive) in the context of problems that compare areas and distances of plane figures. We create the instructional design considering the conceptual goals for the students. The design of lessons (itineraries of problems) is based on the researchers’ mathematical understanding and the researcher’s hypotheses about student’s cognitive characteristics. We have this information from the teacher of the students participating in the experiment, and from the first problem proposed to the students, which is considered as a priori test. This problem (root problem) shares common resolution strategies with the subsequent problems. It also shares conceptual and procedural knowledge. As Simon (1995) states, “the teacher has no direct access to student’s knowledge. He must infer the nature of the student’s understanding from his interpretation of his student’s behaviours, based on his own schemata with respect to mathematics, learning, students, and so on.”(Simon, 1995, p.135).

The researcher’s learning goals provides a direction for a hypothetical learning trajectory. Simon (1995) uses the term trajectory to the teacher’s (researcher in our case) prediction of the path by which learning may proceed. It is hypothetical as the researcher does not know the learning trajectory in advance. We use the term actual learning trajectories (Leikin & Dinur, 2003) to refer to the learning routes that students actually follow in the context of the implementation of the instructional sequences. In our case, the global hypothetical learning trajectory is defined by the design of different itineraries of problems that the students can follow during the teaching experiments. We design three hypothetical itineraries of problems from the consideration of an initial root problem. Depending on the necessities of each student and their resolution process in the first problem, we assign similar problems of increasing level or we propose that the students solve other problems that allow them to overcome the difficulties encountered. The students considered in this research are high achieving students that followed the third itinerary, which is based on problems of level three (levels are defined in previous section). We analyse in depth the hypothetical learning trajectories of three prototypic students in the fourth chapter.

We define, in the sense of Simon (1995), each problem as a micro-cycle of the hypothetical learning trajectory (problems designed for the full teaching experiment), which can also be considered as hypothetical learning trajectories. Thus for each
problem, we consider the researcher’s learning goal, the researcher’s plan for the teaching experiment and the researcher’s hypothesis of the learning process.

In this study, we focus on four consecutive problems solved by the students during the teaching experiment, which consist in four lessons of one hour. Each problem is also a hypothetical learning trajectory. We design for each problem the basic space of the problem (section 2.1 Cognitive approach), the concepts and procedures involved in the different resolution strategies and the environment in which the problem is solved (only paper-and-pencil or technological environment). The basic space of the problem helps to identify the starting level of understanding. We also design the pedagogical space which consists in the messages that the tutor (or teacher) can give to the students to help them during the resolution process. The tutor can help the students by following a protocol detailed in the third chapter. We take into account the tutor’s characteristics, as he can give messages not prepared a priori in the case where students encounter unexpected difficulties. The pedagogic space of the problem helps to achieve the end goal. For each problem, we detail the initial expectations concerning the problem, and after the teaching experiment we analyse the resolution process and the effect it has on the student, taking into account the problem, the tutor’s interactions (orchestration) and the synergy of environments (paper-and-pencil and GeoGebra). We characterize this micro-cycle in terms of the effect of the teaching experiment on the student’s geometric competences.

Figure 2.6: Mathematics teaching cycle (Simon, 1995)
We define transitions between each micro-cycle (problem) in terms of geometric competences and reflect on the transitions to obtain feedback of each micro-cycle (Figure 2.7). A main goal of the research is to achieve a close matching between hypothetical and actual learning trajectories, in the context of the proposed tasks and the geometrical competences considered.

**Figure 2.7:** Local cycle (itinerary) formed by four micro-cycles

### 2.4 The thematic approach: Euclidean area and similarity theory

**Euclidean geometry and the concept of distance**

The purpose of this section is to give conceptual and methodological foundations to the problems which constitute the instructional design (ID). We desire in some sense a coherent set of problems from the point of view of mathematical concepts, i.e., a self-contained bag of problems. Two core mathematical concepts play a main role in the instructional design. Both concepts, as we shall see in the following lines, appear to be strongly related: similarity theory (ST) and the theory of the existence (and unicity) of a Euclidean area (EA). *As a matter of fact, one of the main theoretical points of the ID is to build a small set of problems which help the students to grasp the deep connections between ST and EA.*

In order to present a clear explanation, it is necessary to describe briefly the idea behind the term *geometry*. This means bearing in mind the formal character of a geometry, as a formal system with its set of axioms and rules of inference. Furthermore, a geometry may be interpreted in models. This suggests that notions such as point, (straight) line, distance function and area function, as well as properties such as to be a point between two points, are in some sense undefined. Euclidean geometry is necessarily linked to the distance $d(.,.)$. From the point of view of formal systems, constant symbols, predicate symbols, sorts and the like are to be interpreted in models in order to get a (relative) meaning. So, $\mathbb{R}^2$ is a model of Euclidean geometry, and particularly a categorical model, i.e., one can prove that any model of Euclidean geometry is isometric to $\mathbb{R}^2$. This fact justifies the usual claim that $\mathbb{R}^2$ is *the* model of Euclidean geometry (modulo isometries). This is
Nonetheless not true of Hyperbolic geometry, which shares with Euclidean geometry all the axioms except the fifth postulate 5P. The models of Hyperbolic geometry, such as abstract 2-dimensional manifolds, have constant Gaussian negative curvature \( K = -\frac{1}{R^2} \) where \( R > 0 \). There are then an uncountable number of non-isometric models of Hyperbolic geometry, since from the celebrated Gauss’ Theorem Egregium, it is well-known that (Gaussian) curvature is an invariant of isometries.

The modern formalization of Euclidean geometry was carried out by Hilbert (1904), who took great pains to elucidate the axioms behind Euclid’s famous elements. One of these was the controversial Fifth Postulate (5P). This was already formulated by Euclid, but other crucial axioms were for the first time presented by Hilbert, such as, the postulate of continuity, which is at the basis of the fact that a line of Euclidean geometry is identifiable with the field of real numbers, i.e., that a line can be identified with the completion of the field of rational numbers.

Let us now have a look at the two fundamental notions of the instructional design, ST and EA.

**On similarity theory (ST)**

Let us define formally the concept of similar triangles. Two triangles ABC and EFG are said to be similar if and only if \(<A = <E, <B = <F and <C = <G. An important goal of this section is to study the SSS similarity criterion, namely that if two triangles are similar, then they are proportional, i.e. \( AB/EF = BC/FG = AC/EG \). Let us remark that the converse also holds. The SSS similarity criterion is usually taken for granted, but it is worth pointing out that this result is far from obvious from a formal point of view. As we shall see later on, the SSS similarity criterion turns out to be a key for the proof of Pythagoras’ theorem and the proof of a fundamental lemma which lies at the very foundations of the existence of a Euclidean area. Furthermore, Pythagoras’ theorem has deep consequences for Euclidean geometry, since the distance of what we call a Euclidean geometry turns out to be equal to the so-called \( d(.,.) \) distance function, which

---

3 Every Cauchy sequence of rational numbers has a limit.

4 The Euclidean distance between two points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) of the plane, denoted \( d_2(P, Q) \) is defined as: 

\[
d_2(P, Q) = \left[ ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{\frac{1}{2}} \right]
\]

We enumerate now some of the properties that characterize this distance function.

1. The distance between two points depends only on the position of one relative to the other; it depends only on the differences \( x_2 - x_1 \) and \( y_2 - y_1 \) of their coordinates. This property (that the distance between two points does not change when both points are shifted by an equal amount in the same direction) is called translation invariance.

2. The distance from a point \( P \) to a point \( Q \) is equal to the distance from a point \( Q \) to a point \( P \): \( d_2(P, Q) = d_2(Q, P) \). This property is called the symmetry of the distance function.

3. The triangle inequality: \( d_2(P, R) \leq d_2(P, Q) + d_2(Q, R) \)

4. The distance \( d_2(P, Q) \) between any two points is nonnegative for all \( P \) and \( Q \). That is, \( d_2(P, Q) \geq 0 \). The sign of equality holds if and only if the points \( P \) and \( Q \) coincide. This is often called the positivity of the distance function; it follows immediately from the definition.
we have already mentioned before. The following result turns out to be a key lemma for the proof of the SSS similarity criterion. We will omit the proof because of some involved technicalities.

Lemma: Let $l_1$, $l_2$, $l_3$ be distinct parallel lines in a Euclidean geometry. Let $t_1$ and $t_2$ be two secant lines which intersect the lines $l_i$ at $A$, $B$, $C$ and $D$, $E$, $F$ with the constraint that $B$ belongs to the segment $[A,C]$ (Figure 2.8). Then: \[
\frac{BC}{AB} = \frac{EF}{DE}.
\]

Figure 2.8: Thales theorem

If $P$ has coordinates $(x, y)$ and $Q$ has coordinates $(ax, ay)$, where $a$ is a nonnegative constant, then $d_2(O,P) = a \cdot d_2(O,Q)$; here $O$ denotes the origin $(0,0)$. This property is sometimes called the homogeneity of the distance function, and it holds because $d_2(O,Q) = a \cdot d_2(O,P)$.

Finally, the Euclidean distance has still another property:

5. The Euclidean distance between two points remains unchanged if the $(x, y)$-plane is rotated about the origin through some angle. This property is called rotation invariance.

It turns out that many other distances can be defined, $d(P, Q)_p$, where $p$ is a real number such that $p \geq 1$. In order to be called a “distance”, a function of $P$ and $Q$ must have the properties 1 through 5. The Euclidean distance $d_2$ alone has all six properties. For instance, we define below the distance $d_1$ that gives rise to the Taxi-Cab geometry: $d_1(P, Q) = |x_2 - x_1| + |y_2 - y_1|$. The taxi-cab metric is well-known. In fact, a whole set of “metrics”, was published by Minkowski (1864-1909). Taxi-Cab geometry has a wide range of applications to problems of urban geometry.

Hilbert (1904) developed a more precise axiomatic system for the Euclidean plane that includes that a line of Euclidean geometry is identifiable with the field of real numbers. As a result, Euclidean geometry has become axiomatically grounded in real analysis. An important consequence of this modern approach is that the Euclidean plane is fundamentally linked to the concept of a distance function. Thus, in order to have a deep understanding of modern Euclidean geometry, students need to understand the role played by distances.
Although we have said that we omit the proof of this lemma, let us show at least the strategy of the proof. What we would like to show is that for every positive real number $\varepsilon$, \[
\frac{BC - EF}{AB - DE} < \frac{1}{\varepsilon}.
\] Because $\mathbb{R}$ is an Arquimedean field, the quantity \[
\left| \frac{BC - EF}{AB - DE} \right|
\] is then equal to 0, thus \[
\frac{BC}{AB} = \frac{EF}{DE}.
\]

This lemma depends crucially on the fifth postulate 5P. In fact, as we will briefly survey later, the 5P appears disguised in many situations in Euclidean geometry.

We now proceed to deepen our understanding of the consequences of the previous lemma for similarity theory. One consequence is what we know as Thales theorem, i.e., given a triangle $DEF$ in a Euclidean geometry, if $G$ and $H$ belong respectively to the segments $DE$ and $DF$ and line $GH \parallel EF$, then the following equality holds:

\[
\frac{DH}{DF} = \frac{HF}{DG}.
\]

We abbreviate the proof of Thales theorem. Let $l_1 = (EF)$, $l_2 = (GH)$ and $l_3$ the line through $D$ parallel to $l_2$. By the previous lemma with $t_1 = (DE)$ and $t_2 = (DF)$, we have \[
\frac{GE}{DG} = \frac{HF}{DH} \quad \text{(Figure 2.9)}.
\]

Since $DE = DG + GE$ and $DF = DH + HF$,

\[
\frac{DE}{DG} = \frac{DG + GE}{DG} = 1 + \frac{GE}{DG} = 1 + \frac{HF}{DH} = \frac{DH + HF}{DH} = \frac{DF}{DH}.
\]

So that,

\[
\frac{DE}{DG} = \frac{DF}{DH} \quad \text{or} \quad \frac{DG}{DE} = \frac{DH}{DF}.
\]

Let us consider now two similar triangles $ABC$ and $DEF$. As we are dealing with homogeneous geometry, we can move by a Euclidean transformation the triangle $ABC$ in such a way that there exist points $G$ and $H$ such that the triangle $DGH$ is similar to $ABC$ (Figure 2.10). Now, we are in a standard Thales configuration which immediately gives
the equality of ratios. Going back then to the initial similar triangles ABC and DEF, we are done.

![Transformation of the triangle ABC in the triangle DEF](image)

**Figure 2.10:** Transformation of the triangle ABC in the triangle DEF

We have seen the SSS criterion. We may ask whether the converse holds. The answer is affirmative. For, the similarity theorem (SSS)\(^5\) we have:

In a Euclidean geometry, a triangle ABC is similar to a triangle DEF if and only if

\[
\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}
\]

We move now on the following strong result: similarity theory entails Pythagoras theorem. Consider ABC an arbitrary triangle. We want to prove that ABC is a right triangle at B if and only if

\[
AB^2 + BC^2 = AC^2 \quad (1)
\]

We start proving the *only if* direction.

First suppose that \(<B\) is a right angle of the triangle ABC and D is the foot of the altitude from B. The triangles ADB, ABC and BDC are similar. Thus,

\[
\frac{AB}{AD} = \frac{AC}{AB} \quad \text{and} \quad \frac{BC}{DC} = \frac{AC}{BC}
\]

Hence \((AB)^2 = (AC)(AD)\) and \((BC)^2 = (AC)(DC)\). Now AC = AD + DC so that

\[
(AC)^2 = (AC)(AD + DC)
\]

\[
= (AC)(AD) + (AC)(DC)
\]

\[
= (AB)^2 + (BC)^2
\]

Let us see now the *if* direction.

Now suppose that the triangle ABC satisfies the equation (1). We must show that B is a right angle. Let the PQR a right triangle with right angle at B and \(\overline{PQ} \cong \overline{AB}, \overline{QR} \cong \overline{BC}\). Since the angle \(<Q\) is a right angle we may apply equation (1) to obtain:

\[
(PR)^2 = (PQ)^2 + (QR)^2 = (AB)^2 + (BC)^2 = (AC)^2
\]

Hence, \(\overline{PR} \cong \overline{AC}\) and PQR \(\cong\) ABC by SSS. But this means that \(<B \cong <Q\) is a right angle.

We would like to know whether from Pythagoras theorem we can deduce the similarity criterion (and in fact the existence of similarity theory). Again, the answer is positive\(^6\).

\(^5\) See p. 225, theorem 9.2.6 (Millman and Parker, 1991)
We now turn our attention to a very interesting consequence of similarity theory which is at the heart of the definition of a Euclidean area function (see next subsection). Let ABC be a triangle. Let D be the foot of the altitude from A and let E be the foot of the altitude from B. Consider the following quantities \( AD \cdot BC \) and \( BE \cdot AC \). We realize that these quantities are twice the area of the triangle ABC. But we are studying ST and we do not assume for the moment any definition of the area function. This is not obvious, for we are assuming as a primitive no Euclidean area function.

Let us see now the proof of the equality of the quantities \( AD \cdot BC \) and \( BE \cdot AC \).

If \( \angle C \) is a right angle then \( E = C = D \) and the result is trivial. Suppose on the contrary that \( \angle C \) is not a right angle, then \( E \) is different from \( C \) and \( D \) is different from \( C \). Since the triangles BEC and ADC are similar, we have then by application of the SSS similarity criterion that \( BC/AC = BE/AD \), whence \( AD \cdot BC = BE \cdot AC \)

Before proceeding to the following subsection, let us list without proof some equivalent formulations of the fifth postulate 5P. Some of the results are striking:
- The existence of similar triangles
- The existence of squares (in hyperbolic geometry they do not exist)

The existence and unicity of a Euclidean area function (EA)
In the previous subsection we did not assume any given Euclidean area function. We want now to define the existence and unicity of a Euclidean area function. Before going on to the construction of the area function, let us define it formally:

**Definition:** In a geometry (not necessarily assuming the 5P), an area function is a function \( \sigma \) from the set of polygonal \( \mathcal{R} \) regions to the real numbers \( \mathbb{R} \) such that:

i) \( \sigma(R) > 0 \) for every region \( R \in \mathcal{R} \) (positivity)

ii) If \( \triangle ABC \cong \triangle DEF \) then \( \sigma(\triangle ABC) = \sigma(\triangle DEF) \) (equivalence)

iii) If \( R_1 \) and \( R_2 \) are two polygonal regions whose intersection contains only boundary points of \( R_1 \) and \( R_2 \) then:

\[
\sigma(R_1 \cup R_2) = \sigma(R_1) + \sigma(R_2) \] (complementary area)

iv) If \( R \) is the convex polygonal region determined by a square whose sides have length \( a \), then \( \sigma(R) = a^2 \)

Before dealing with the construction of a Euclidean area function, we sketch some easy properties assuming its existence:

- Rectangles (of length \( a \) and width \( b \)) as expected have (Euclidean) area \( ab \).
- A triangle ABC with altitude AD, has area \( \frac{BC \cdot AD}{2} \)

Both properties are easily deduced using the (ii), (iii) and (iv) axioms of an area function. For the first property, as the area of a square (which exists in a Euclidean geometry!) is equal to \( a^2 \) (if \( a \) is the length of the sides of the square). Consider the following figure:

---

6 At the end of this section we list some equivalent formulations of the fifth postulate 5P.
7 We insist that the properties we state of the existence of a Euclidean area function are far from being obvious. Later on, we see how similarity theory ST allows a coherent definition of the area function.
By (iv) axiom we know that the two inner squares of the square have respectively area $a^2$ and $b^2$. The remaining rectangles are both congruent and hence, by axiom (ii), they have the same area which we denote by $x$. By axiom (iv) the whole square has area $(a+b)^2 = a^2 + b^2 + 2ab$. We are done, since we deduce from this that $x = ab$.

The second property (the area of an arbitrary triangle) follows from extensive use of the previous result and the axiom (iii). As the area of an arbitrary rectangle is the product of its width and length we deduce by axiom (ii) and (iii) that the area of a rectangle triangle is $\frac{1}{2}ab$, where $a$ and $b$ are the sides of triangle (Figure 2.12). Finally, if we are given an arbitrary triangle (Figure 2.13), one can consider a trapezoid (Figure 2.14) by using three times axiom (iii).

The construction of an area function assuming a Euclidean geometry
As we have seen before, ST can be built in the context of a Euclidean geometry. We see now how ST allows us to define an area function successfully. The steps at the heart of
the construction of an area function are the following. We first define the area of triangles which are the simplest polygons. The quantity assigned as area to the triangle must be well-defined. For, given a triangle \( ABC \), one can consider a base side and its so-called altitude, say \( AB \) and \( h \). We define then the area as \( \frac{AB \cdot h}{2} \). But we could have considered the other two base sides and its corresponding altitudes. By similarity theory (see previous subsection), we know that all these quantities are equal. This guarantees that the function we want as area function is well defined.

Once we have defined our candidate function \( A(.) \) (as a area function) for triangles, we have to define the quantity of \( A(.) \) for arbitrary polygonal regions. This is done as follows. We consider a (finite) triangulation \( \{T_1, \ldots, T_n\} \) of a polygonal region \( R \) and we define \( A(R) \) as the sum of all \( T_i \). Again, we have to check that this step in the construction of the area function candidate \( A(.) \) is well-defined. It turns out to be true, but its proof is involves many technicalities. We will therefore omit it.

We already have a function \( A(.) \) which satisfies the properties of an area function. This is not obvious, and its proof is strongly based on similarity theory ST. We ask whether \( A(.) \) is unique. The answer of course is affirmative. Let \( A'(.) \) be another area function. By the third axiom of area, the area of a polygonal region is the sum of the areas of the triangular regions in any triangulation. Hence if \( A'(.) \) agrees with our \( A(.) \), it follows that \( A'(.) = A(.) \). But as we saw, any area function in a Euclidean geometry assigns to a given triangle the quantity \( \frac{1}{2} \) (base side). (altitude).

**Interrelation between similarity theory and the existence of an area function.**

In the previous subsection we have seen how to define the existence and unicity of a Euclidean area function. All we were assuming was similarity theory ST (in the context of a Euclidean geometry). We now see the converse:

**From EA to ST: Euclid’s proof of Thales theorem.**

The area function can be used to prove the basic theorem of similarity theory ST. This approach, which was carried out by Euclid is simpler than the construction of a Euclidean area function (see previous subsection).

---

\(^{8}\) We assume that the polygonal region admits a triangulation.
From the existence of Euclidean area to similarity theory: Euclid’s proof

(BC) // (EF) thus:

\[ \text{Area}(BEF) = \text{Area}(CEF) = \frac{EF \cdot h}{2} \]

Area\((ABF) = \text{Area}(AEF) - \text{Area}(BEF)\)

Area\((ACE) = \text{Area}(AEF) - \text{Area}(CEF) = \text{Area}(BEF) = \text{Area}(CEF)\)

Thus, \(\text{Area}(ABF) = \text{Area}(ACE)\)

\[ \text{Area}(ABF) = \frac{AB \cdot d}{2}, \quad \text{Area}(AEF) = \frac{AE \cdot d}{2} \]

Thus, \(\frac{\text{Area}(ABF)}{\text{Area}(AEF)} = \frac{AB}{AE}\)

But, \(\text{Area}(ABF) = \text{Area}(ACE)\), thus \(\frac{\text{Area}(ABF)}{\text{Area}(AEF)} = \frac{\text{Area}(ACE)}{\text{Area}(AEF)}\).

\(\frac{AB}{AE} = \frac{AC}{AF}\)

Table 2.15: Euclid’s proof
Instructional design problems: Small set of problems which help the students to grasp the deep connections between ST and EA.

Our goal is to help the students to grasp the deep connection between ST and EA and to develop students’ geometrical competences (visual, structural, instrumental and deductive) in the context of the problems considered. Some of the mathematical contents related to the problems considered are:
- expressions for the areas of plane figures
- metric relations in the right-angled triangles
- Thales theorem and its reciprocal
- relations between angles defined by a transversal line secant to parallel lines
- trigonometric relations
- elements and properties of polygons
- congruence of polygons
- similarity of polygons
- equivalence of polygons
- relations between the concepts of congruence, similarity and equivalence
- criteria of congruence and similarity
- relation between the linear elements and the area of similar figures
- basic analytic geometry: line equation, intersections of lines, resolution of linear system of equations, etc.
- concept of parameter

As part of the procedures involved in these problems we identify:
- identifying and representing the heights of triangles
- applying formulas for the area
- identifying a reference system
- obtaining a line equation
- applying similarity ratio to compare areas
- applying Thales theorem and its reciprocal
- applying congruence, similarity criteria
- applying properties of the area function as a measure function
- transforming figures in equivalent figures

We expect that the students will obtain a deeper understanding of these geometric concepts and will acquire geometric competences as a result of the proposed tasks, the tutor’s orchestration and the synergy of environments. The problems considered can be solved by combining visual and deductive approaches. The use of GeoGebra may help the students to visualize, explore, conjecture and understand the structure of the problem. As stated by Laborde, Kynigos, Hollebrands and Strässer (2006), “researchers and educators stressed the importance of the role of visualization in a geometry activity: Solving a geometry problem goes beyond the visual recognition of spatial relations. It is commonly assumed that the teaching of geometry should contribute to the learning of: (1) The distinction between spatial graphical relations and geometrical relations, (2) The movement between theoretical objects and their spatial representation, (3) The recognition of geometrical relations in a diagram, (4) The ability to imagine all possible diagrams attached to a geometrical object. The second kind of ability is particularly
critical in the solving processes of students faced with geometry problems requiring exploration in which a cycle of interpreting, conjecturing, and proving may take place because of this flexibility between spatial representations and theoretical knowledge. Such assumptions about the teaching and learning of geometry have led some researchers to focus on the role of graphical representations provided by the software” (Laborde et al., 2006, p.277).

As detailed in the third chapter (Design research and methodology), after analysing the resolution process, for each problem we reflect on the findings in terms of the effect they have on students’ geometric competences and we formulate the feed-forward. After the analysis of each micro-cycle, we describe the transitions for each competence between consecutive problems. We develop a coding system adapting Drijvers’s system (2003). As Drijvers (2003) states, “this coding system guaranteed the links between HLT and key items in the teaching materials and was a means of verifying whether the intentions in the HLT were indeed realized in the teaching materials” (Drijvers, 2003, p.26). The (+++) scores refer to observations that reveal a deep insight into the competence; the (+) scores refer to observations that reveal the expected insight into the competence; the (o) scores refer to insights into the competence that are not completely correct but do contain valuable elements; and the (-) scores refer to insights into the competence that are not adequate. We elaborate distinctive attributes for this coding system concerning each competence in the third chapter. Several researchers have described processes central to geometrical thinking that have distinctive attributes across a number of Van Hiele levels (Gutiérrez, Jaime & Fortuny 1991; De Villiers, 1999). The Van Hiele model describes the geometric thinking in terms of five hierarchical levels. We adapt these distinctive attributes to elaborate a coding system. We present in detail the indicators for the different scores in the third chapter (3. Research design and methodology). The coding system which has been revised by other researchers of the group, allows us to define the transitions between problems in terms of geometric competences for each student. We characterise the learning trajectories in terms of these transitions. Nevertheless, this coding system can not be extrapolated to other contexts such as different problems. We use them to define a unified criterion for describing transitions in students’ competences. At the end of the teaching experiment, we report the transitions between all the micro-cycles. We reflect on the expectations that we had at the start of the teaching experiment and we formulate the feed-forward of the cycle.
3. Research design and methodology

In this chapter we explain the design research and the methodological aspects of the study. The first phase of the research consists in the development of a Hypothetical Learning Trajectory as defined in the theoretical framework (Simon, 1995). We consider the design of instructional activities which are based on problems related to the concepts of distances and areas of plane figures. As stated by Drijvers (2003), one important feature of the design research is the adaptation of the learning trajectory throughout the research. The instructional sequence and the teaching experiment conditions are adjusted on the base of previous experience. This iterative feature (research cycle of design, implementation, analysis and refinement) allow us to adapt the instructional sequence to obtain a closer matching between the hypothetical learning trajectory and the actual learning trajectory (Stylianides & Stylianides, 2009). We also assume that design research is appropriate for our study due to the kind of research questions considered:

- How can the use of GeoGebra be integrated into a teaching sequence to promote students’ geometric competences (visual, structural, instrumental and deductive)?

- What are the students’ behaviours when solving problems under the influence of the instructional activities, the teacher’s orchestration and the synergy of paper-and-pencil and GeoGebra?

We expect to answer the first question by designing a Hypothetical Learning Trajectory. The use of design research is adequate because considering the integration of GeoGebra in learning requires the adaptation of mathematical tasks. For instance, Laborde (2001) points to a classification of mathematical tasks according to the role that dynamic geometry software plays in it. She distinguishes between four types of roles. First, the software could facilitate material aspects of the tasks without changing it conceptually. Secondly the software could be used as a visual amplifier (Pea, 1985) in order to facilitate observations, such as for example identifying properties of geometric figures. Thirdly, by providing special tools the software allows students to solve mathematical tasks in new ways. Finally, dynamic geometry software allows for the creation of a new type of mathematical problems that could not be treated in the classroom without technology, such as dynamic investigations of mathematical concepts. We consider the development of tasks of the second and third type to achieve our research goals. As stated by Laborde, Kynigos, Hollebrands and Strässer (2006), “in the last two types, tasks are changed in some way by the mediation of DGE, either because the solving strategy differ from what they usually were, or because they simply are not possible outside DGE” (Laborde et al., 2006, p. 293). We expect to answer the second question by following an observational methodology by analysing observable behaviours (interactions between the different agents). We follow a case study and analyse it from a qualitative perspective. In the following sections we describe first the design research and then the methodology.

3.1 The design research: the instructional design and the concept of HLT

In this section we explain the pedagogical motivations underlying the so called instructional design (ID). We consider the design process as an integrated part of the research. The pedagogical motivations turn out to be a set of constraints of different
nature faced by the researchers in the task of conceiving the ID. In terms of mathematical knowledge, we need to focus on a specific topic, and in our case, we have decided to centre on problems that compare areas and distances of plane figures. Our point of departure is the kind of problems defined by Cobo (1998), which considered problems based only on the comparison of areas of plane figures. We extend Cobo’s (1998) approach with problems based on similarity theory. The main goals of this extension are manifold. On the one hand, we want to improve the students’ understanding of the deep relation of Euclidean area and similarity theory (see section 2.4 the thematic approach). On the other hand, the kind of problems we have proposed are aimed at targeting the more important and standard features that GeoGebra (GGB) offers to the students, such as the dynamic component (see section 2.1 the instrumental approach).

In the conception of the ID, we carefully selected a small set of problems out of an initial larger set of problems. We required that the problems satisfied a series of constraints, enumerated as follows: to share a close logical structure (see section 2.2 the cognitive approach), to have in common resolution approaches and hence similar basic spaces, and finally to be a priori suitable for the use of GeoGebra.

With respect to the logical structure (section 2.2 The cognitive approach), we have considered problems with at most two nested universal quantifiers, which ask the students to find the relation between areas or segments of plane figures. Most of the problems have ‘find a’ questions, i.e., statements which have a meta placeholder. A prototypical example would be phrases such as ‘find a relation between the areas $A_1$ and $A_2$’, where the student is requested to prove an underspecified statement. In our example, the students have to guess/conjecture a relation between $A_1$ and $A_2$ (see Tao’s (2006) classification of types of problems) and to prove it. As it may be noticed, this kind of ‘find a’ statement has been strategically considered in order to put in scene the dynamic character of software of dynamic geometry (in our case GGB), which we expect will help students to improve their geometrical competences through the synergy of paper-and-pencil and GeoGebra environments.

Concerning the resolution strategies, the problems of the ID should satisfy another constraint, namely to be similar in terms of basic space. They turn out to share the resolution strategies, such as equivalence of areas due to equicomplementary dissection rules, application of formulas (area of figures), particularization, etc. The problems share different solving strategies and can be solved by mixing graphical and deductive issues.

Regarding the suitability of the problems to be solved with GGB, we have decided to focus on the second and third types of roles that DGS plays in the problems (Laborde, 2001): a) to facilitate the observation of geometrical properties or invariants, and b) to allow the students to cope with problems in a new way by providing new tools (especially the dragging tool combined with the simultaneous use of the geometric and the algebraic windows).

The set of problems of the ID has been designed in such a way as to have a hierarchical structure that allows the students to follow different learning routes based on the resolutions of certain problems which we call itineraries of problems. The topology of the ID (seen as a graph -see figure 3.1-) consists of a rooted graph which is almost a rooted tree but, as we shall see later, it can have cycles. The root of our graph of
problems is what we call a root problem. A root problem plays a special role in the design of itineraries of problems, namely it has three important features. Its logical structure is shared with the other problems; it has a medium level of complexity (level two defined in section 2.2 The cognitive approach), and the set of a priori solving strategies contains the main types of problem solving strategies in the subsequent problems. In the case of problems that compare distances and areas of plane figures, we choose a root problem that shares the following resolution strategies: applying equicomplementary dissection rules to compare areas (properties of the area function as a measure function), obtaining equality of ratios to compare linear elements of the figure (similarity theory e.g. Thales theorem, similarity criteria, and dilatations).

The itineraries of problems that compare areas and distances in plane figures are designed from the consideration of the root problem, which is the first problem set to the students for solving with paper-and-pencil. As Drijvers says, “the Hypothetical Learning Trajectory represents a learning route that has different itineraries and students can go through it at different speeds” (Drijvers, 2003, p.24). We use the Hypothetical Learning Trajectory for designing and planning short teaching cycles of four lessons. We consider each problem as a micro-cycle of the learning trajectory. For each problem we analyse the expected effect of the problem and the effect it has on the student through the different geometric competences. We shall report the transitions in the acquisition degrees of geometric competences between the problems solved by a student.

We proceed now to characterise the three main itineraries of the ID. The selected root problem corresponds to Euclid’s 43 rd proposition of the Elements (about 300 B.C., Euclid of Alexandria wrote the treatise in thirteen books called the Elements). We adapt this proposition and we define it as the root problem. In the a priori analysis of this problem, we identify three main resolution approaches: the approach based on equicomplementary dissection rules of areas, the approach based on obtaining equality of ratios to obtain the area of each rectangle applying area formulas, and the approach based on considering analytic geometry. The selected problems share these strategies, and we define the itineraries in terms of the basic space of the root problem (Figure 3.1) and the difficulty levels of the problems. As mentioned before, there are different difficulty levels for the problems of the ID. We define three difficulty levels in terms of the logical structure of the problem and the different resolution approaches defined in the basic space (see section 2.2 The cognitive approach). Problems of level one are problems that have only one step in the main resolution strategies, and they can be considered as tools for solving the problems of higher levels. For instance, a problem of level one is: ‘The median of a triangle splits the triangle into two triangles. What is the relation between the areas of the inner triangles?’ The resolution strategy is based on comparing the base and the height of both triangles. The problems of level two and level three are similar in terms of basic space, but the strategies based on equicomplementary dissection rules are easier to visualize for problems of level two. For instance, in the case of problems of level two, we can split the figure into congruent figures whereas for problems of level three the figures to compare are not congruent. We can extract equivalent figures (figures of the same area that are not congruent) or similar figures. Moreover, the logical structure of problems of level three has two nested universal quantifiers. As we plan to analyse the connectivity and synergy between paper-and-pencil and GeoGebra environments, we need to consider problems to be solved only with paper-and-pencil and problems to be
solved in a technological environment (GeoGebra and paper-and-pencil) in each itinerary. We then incorporate into each itinerary pairs of similar problems to be solved respectively only with paper-and-pencil or in both environments (paper-and-pencil and GGB environments). It is then expected to observe and understand the synergy of both environments. We explain now how the students are assigned itineraries from the root problem (Figure 3.1).

- In case the students are not able to solve the root problem, they follow the first itinerary, which mainly consists of problems of level one that can be considered as ‘tools’ (e.g. decomposing the figure, comparing base and heights to apply the area formula of the triangle, applying Thales theorem to obtain equality of ratios) for solving the problems of level two and level three. At the end of the first itinerary there are two problems of level two. If the students solve these problems we set the root problem again and they can be derived to other itineraries. Thus, the objective of the first itinerary is to help the students to become acquainted with the mathematical concepts and procedures as a toolbox for the following problems.

- If students solve the root problem using a strategy based on equicomplementary dissection rules we propose that they follow the third itinerary. The third itinerary is based on problems that also can be solved by applying equivalent dissection rules, but these problems have a higher level of difficulty (level three). Moreover, we introduce problems in which the students have to consider other strategies than the first one, such as considering equality of ratios for instance, the problem of the scaled triangles (section 3.1.2.2 The scaled triangles problem).

- If the students solve the root problem, which is a problem of level two, without considering the strategy based on equicomplementary dissection rules, we propose that they follow the second itinerary. The second itinerary is based on three problems of level two and two problems of level three. These problems foster the consideration of strategies based on equicomplementary dissection rules of areas. At the end of the second itinerary, if the students solve the problems of level three, they continue with the third itinerary (shift of itinerary).
Each lesson is orchestrated by the tutor, who guides the students in the problem solving process by giving some messages and setting new problems. For each problem we consider the pedagogical space (Cobo, Fortuny, Puertas & Richard, 2007), which consists in cognitive messages and contextual messages that are defined a priori to help the students in the resolution process. We consider the tutor’s orchestration (Trouche, 2004) as the messages defined in the pedagogical space and also as the ad hoc decisions taken during the teaching experiment. For instance, we consider dealing with unexpected aspects of the technological tool or the mathematical tasks. As stated by Drijvers, Doorman, Boon and Van Gisbergen. (2009), “many studies focus on the students’ instrumental genesis and its possible benefits for learning (e.g., see Kieran & Drijvers, 2006). However it was acknowledged that instrumental genesis needs to be guided, monitored and orchestrated by the teacher” (Drijvers et al., 2009, p. 145). In our study, we include the tutor’s orchestration in the instructional design. A researcher of the group involved in this research has the role of a tutor who guides the students during the teaching experiment. We consider that the autonomy of the tutor should be limited. This
inquiry based approach to the lessons leads the students to accept the responsibility for the development of the task. The tutor fosters students’ autonomy by intervening only at certain moments, as we show in the following section (3.1.1 The role of the tutor). The tutor should not solve the problem for the students with the messages, but students should not get lost in a particular task. For this reason we consider different itineraries of problems that allow the students to go through the “learning route at different speeds” (Drijvers, 2003). The objective of the learning route is to help the students to improve their geometrical competences as well as providing insight into the interrelation between similarity theory and area.

We consider each problem as a micro-cycle of the local learning trajectory (one itinerary) of the global instructional design which consists of all the itineraries. As part of the hypothetical learning trajectory, we consider the tutor’s orchestration. In the following section, we specify the role of the tutor and explain how the tutor guides the students.

3.1.1 The role of the tutor

In this section we describe the role of the tutor in the teaching experiment. This is a relevant aspect since the tutor can take ad hoc decisions during the teaching experiment. We try to minimize the autonomy (ad hoc decisions) of the tutor and we expect the students to solve the problems on their own, explaining in detail their resolution strategies.

The tutor has all the information about messages and strategies (pedagogic space of the problem) and he tries to be systematic when giving the messages. He also assigns the problem to the students while explaining that they can request help. Moreover, on the students’ worksheet there are the following indications: ‘El tutor et pot donar tres tipus d’ajudes: ajudes sobre la comprensió de l’enunciat, ajudes per a la resolució del problema i ajudes sobre la comprovació de la solució’. For the problems to be solved with GeoGebra, there is also the following message: ‘el tutor et pot donar ajudes sobre l’ús de GeoGebra’.

The criteria for assigning problems are established a priori in the instructional design. The tutor follows two criteria for giving messages to the students, which correspond to the following situations: the student requests help from the tutor; the student does not request help from the tutor. In the first case, when a student requests help, the tutor identifies which phase of the resolution process the student is in. Then he searches in the messages list for the most appropriate message and he tries to give first a message of level one. We understand that the message is appropriate if it answers the student’s question and gives the minimal information. For instance, in the familiarization phase, as the concepts involved in the problems’ statement are simple, the questions asked by the students are usually related to the distinction between the drawing and the figure (what properties can be considered as givens of the problem). In this case, he selects a message of level one from the list to make the student read more carefully the statement of the problem. Other questions are related to the logical structure of the problem and the use of GeoGebra. We assign to these questions messages of increasing level if the student does not react to the previous message. If the tutor does not identify messages on the list for the student’s question, he tries to answer with a message to guide the student, but always trying to give as little information as possible. In this case, he also identifies the
resolution phase and the resolution approach considered by the student. For instance, one of the students asked: ‘No pot ser això, és massa difícil. Els altres alumnes han resolt aquest problema? Vaig bé?’ As the tutor recognizes actions of a resolution strategy of the basic space (strategy based on equality of ratios to obtain the area of the rectangles) and he observes that the student is not lost he decides that the student does not need further information. He answers: ‘Hi ha diverses maneres de resoldre aquest problema. Has trobat resultats que et poden ser útils.’

In the second case, when the student does not request the help from the tutor, the tutor tries also to follow a systematic criterion for giving messages, based on giving as little information as possible to guide the student in the resolution process. The tutor observes the student during the familiarization phase. He observes the student’s actions (graphic actions, inferences) to check that he has understood the statement of the problem. He gives messages of level one if he detects that the student does not understand the statement of the problem. For instance, some students make mistakes when labelling the points, misunderstand the question of the problem, add restrictions, etc. The tutor suggests that the students read the statement of the problem again. If the student does not react, he gives more information (message of level two or three).

During the resolution process, if the student is not active (there is no feedback such as graphic actions on the screen, on the worksheet, written work, etc), the tutor observes the resolution process and the last step in the current strategy and he gives the student a message related to the last step. When all the messages related to the step have been given, he considers messages to foster a change of strategy and as a last resort he proposes a change of itinerary (first problem of the previous itinerary).

In the verification phase, the tutor observes the resolution strategy by looking for recognized actions. If there are steps not justified deductively, the tutor encourages the student to justify the step deductively by giving him messages to foster the student’s awareness of the necessity of proof or counter-proof. For instance, a student asserts the following statement which is false: ‘Com els rectangles tenen tots els costats paral·lels (AAAA in analogy with the criteria AAA for triangles) llavors són semblants’. In this case, the tutor tries to encourage the student to check the validity of the statement and he asks the student: ‘Estàs segur?’ As the student does not react to the message (he answers ‘Si, si tots els angles són iguals les figures tenen la mateixa forma’), the tutor then suggests a search for a counter-example (heuristic message of level two). As a last resort, the tutor helps the student to find the counter-example. Finally the student realizes that the rectangles he has considered verify another condition: they share the diagonal. He deduces that he can transform one rectangle into another through dilatation. This observation reveals the expected insight into the structural competence as a result of the student-tutor interactions.

Finally, after the verification phase, the tutor sets the student the next problem and indicates the environment in which the problem has to be solved. For the problems to be solved in a technological environment, the tutor asks the student to construct with GeoGebra the figure associated to the statement of the problem (familiarization phase).
3.1.2 A priori analysis of the selected problems

In this qualitative case study, we focus on the prototypic cases of three high-achieving students that have followed the third itinerary, and we analyse in depth the resolution of four of the problems solved during the teaching experiment: a) the root problem (paper-and-pencil), b) the scaled triangles problem (technological environment), c) the median problem (technological environment) and d) the quadrilateral problem (paper-and-pencil). In this section, we carry out the a priori analysis of these four problems. For each problem we distinguish the basic concepts in the statement of the problem; we characterize the problem’s logical structure, and we examine the different resolution approaches, namely its basic space (Cobo, 1998). These problems share resolution approaches such as: equivalence of areas applying equicomplementary dissection rules; obtaining equality of ratios to compare distances or areas (applying area formula or similarity ratio for areas), and particularization (back and forward strategy based on finding invariants to generalize). We also consider strategies based on modifying the problem slightly. For instance, considering special cases of the problem, such as particular, degenerate cases, or considering equivalent problems which would imply the initial problem. The heuristic strategy of considering particular, degenerate cases can be considered as a prior approach for any resolution strategy. Particularization can shed light on the general solution. Nevertheless, we only include the particularization approach in the basic space of the problem when we obtain a problem which is equivalent or can be generalized to the initial problem (back and forward strategy).

For each resolution approach, we identify the necessary contents, procedures and GeoGebra instrumented techniques (for the GeoGebra problems) to solve the problem. We condense these concepts, procedures and techniques in a table. Finally, we define the a priori cognitive and contextual messages by considering the successive steps of each resolution strategy. Nevertheless, as already mentioned, we design messages that give little information on the contents and procedures.

3.1.2.1 The root problem (paper-and-pencil)

The root problem can be generalized to the case of a parallelogram which corresponds to the 43rd proposition in the first book of Euclid’s elements: “In any parallelogram the complements of the parallelograms located on the diagonal are equal”. The statement of the root problem is the following:

Let E be any point on the diagonal of a rectangle ABCD such that AB = 8 units and AD = 6 units.
The parallel line to the line (AB) through the point E intersects the segment [AD] at the point M and the segment [BC] at the point O.
The parallel line to the line (AD) through the point E intersects the segment [AB] at the point N and the segment [DC] at the point P.
What relation is there between the areas of the rectangles NEOB and MEPD in the figure below?
The objective of the problem is clear for the students, but there is a ‘find’ question which requires finding a relation between the areas of both rectangles. We identify in the problem’s statement the concepts of rectangle, diagonal of a rectangle, parallelism and areas of rectangles. There is a labelled figure inserted in a grid in the statement to facilitate students’ understanding of the root problem. This a priori decision may obstruct the understanding of the structure of the problem, as we have observed in the analysis phase. Some students focus on the fact that the point E has a concrete position on the diagonal and they do not pay attention to the verbal statement that introduces E as any point of the diagonal.

We show the logical structure of the root problem in the table below (Table 3.1): there is one nested quantified box (the property to prove is in the second box), there is a free object (the point E on the diagonal), whereas the outside rectangle ABCD is fixed. The students have to conjecture the relation \( \Phi((\text{MEPD}, (\text{NEOB})) \) between the areas of both inner rectangles. We give the concrete lengths of the rectangle’s sides and we insert a figure with a grid on the statement of the problem to facilitate students’ understanding of the root problem.

\[
\forall E
\]
\[
E \in AC \land \text{parallels through } E \text{ to the sides of the rectangle ABCD} \rightarrow \\
\rightarrow \Phi((\text{MEPD}, (\text{NEOB}))
\]

**Table 3.1:** Logical structure of the root problem (it corresponds to the verbal statement presented in the students’ worksheet)

Another reason for considering a concrete rectangle is that we try to avoid influencing students’ strategies. We conjecture that considering any rectangle may foster a strategy based on visualizing the inner congruent triangles and applying complementary dissection rules of areas. Instead, if we had given a concrete position of the point E on the
diagonal (i.e., the length of the segment AE) we would have fostered algebraic strategies based on obtaining the concrete values of both areas. The logical structure of this problem does not foster an algebraic resolution, as the students can not obtain the concrete values of the areas. For instance, a “geometric” student (Kruteskii, 1976) may be willing to solve this problem using visual methods that allow him to discern congruent triangles.

With these decisions, we tried to avoid giving information to the students about a particular strategy. For instance, if we had introduced the coordinates of the vertices, the students may have attempted to use analytic geometry. We tried to avoid focusing students’ attention on a particular resolution approach because this problem is considered as a pre-test. In the instructional design, the problems are progressively introduced in a more generic way (without figures in the statement and without concrete data about the lengths of the elements). In the following section we describe different approaches to solving this problem. We exemplify these approaches with some resolution strategies and define the basic space of the problem.

a) Basic space of the root problem

We distinguish three resolution approaches. The first approach is based on equicomplementary dissection rules of areas, the second approach is based on obtaining equality of ratios to obtain the area of each rectangle applying area formulas, and the third approach is based on considering analytic geometry.

1. Resolution approach: Decomposition of the rectangle into congruent triangles

It requires identifying the inner triangles AEM, EPC and ADC and the congruent triangles. The diagonal splits each inner rectangle into two congruent triangles, thus we have the following equalities of triangle areas:

\[
(ADC) = (ACB), (AEM) = (AEN) \text{ and } (EPC) = (EOC)
\]

On applying equicomplementary dissection rules, we get the equality of areas:

\[
(MEPD) = (ADC) - (AEM) - (EPC) = (ACB) - (AEN) - (EOC) = (NBOE)
\]

2. Resolution approach: equality of ratios

There are many approaches for this strategy. All these approaches are based on applying the formula of the area of a rectangle.

- Applying Thales theorem to obtain the ratio between the homolog segments of the triangles AEM and EOC. We obtain:

\[
\frac{AM}{OC} = \frac{EM}{OE} \Rightarrow AM \times OE = OC \times EM \text{ and thus the equality of areas.}
\]

- The right-angled triangles ACD and AME are similar (AAA criteria). We consider a trigonometric approach: \( \tan(\alpha) = \frac{8}{6} = \frac{ME}{AM} \) (a secant line to two parallel lines defines equal interior angles). Considering that triangles ACD
and EPC are also similar, we obtain the equality $\frac{8}{6} = \frac{8 - ME}{6 - AM}$. This equality can also be derived from the first equality of ratios applying properties of ratios.

We get that the areas of both rectangles are equal: $8(6 - AM) = 6(8 - ME)$

- **Equivalent problem**: areas of the rectangles ABOM and ANPD (Figure 3.2)

  Applying Thales theorem (similar triangles in usual configuration), we obtain the equality of ratios $\frac{AN}{AB} = \frac{AM}{AD}$ and thus the equality of the rectangles' areas.

![Figure 3.2: Equivalent problem](image)

3. **Resolution approach: Analytic geometry**

   This strategy requires placing the origin and the coordinate axes appropriately. We may also modify slightly the statement of the problem by considering the other diagonal DB and the resultant equivalent problem.

   The equation of the diagonal is $y = \frac{3}{4}x$. We obtain the coordinates of the point E as $E = (x, \frac{3}{4}x)$. By applying the formula for the area of the rectangle, we obtain the areas in terms of the coordinates of the point E.

   \[ A_1 = x(6 - \frac{3}{4}x) = -\frac{3}{4}x^2 + 6x \]

   \[ A_2 = (8 - x)\cdot\frac{3}{4}x = -\frac{3}{4}x^2 + 6x \]

   Thus, $A_1 = A_2$

In the following section we consider the mathematical contents of the problem in terms of necessary concepts and procedures to solve the root-problem considering all the resolution approaches of the problem’s basic space.

**b) Table of contents and procedures**
We condense in the following table the necessary concepts and procedures to solve this problem, considering the different resolution strategies of the problem’s basic space.

<table>
<thead>
<tr>
<th>Mathematical content of the root problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Concepts</strong></td>
</tr>
<tr>
<td>Definitions and elements of figures</td>
</tr>
<tr>
<td>The diagonal of a rectangle splits the rectangle in two congruent triangles</td>
</tr>
<tr>
<td>Congruence criteria of triangles</td>
</tr>
<tr>
<td>Formula for the area of triangle and the area of a rectangle</td>
</tr>
<tr>
<td>Decomposition of areas</td>
</tr>
<tr>
<td>Thales’ theorem</td>
</tr>
<tr>
<td>Trigonometry of the right-angled triangle</td>
</tr>
<tr>
<td>Similarity of triangles and ratio between homolog sides</td>
</tr>
<tr>
<td>Criteria of congruence for triangles</td>
</tr>
<tr>
<td>Coordinate axes</td>
</tr>
<tr>
<td>Straight line equation</td>
</tr>
<tr>
<td>Distance between points</td>
</tr>
<tr>
<td><strong>Procedures</strong></td>
</tr>
<tr>
<td>Applying formulas</td>
</tr>
<tr>
<td>Applying congruence criteria</td>
</tr>
<tr>
<td>Applying trigonometry of the right-angled triangle</td>
</tr>
<tr>
<td>Applying Thales’ theorem</td>
</tr>
<tr>
<td>Applying equivalence of areas due to complementary dissection rules</td>
</tr>
<tr>
<td>Applying similarity criteria</td>
</tr>
<tr>
<td>Decomposing a rectangle</td>
</tr>
<tr>
<td>Choosing the origin and coordinate axes</td>
</tr>
<tr>
<td>Considering particular, degenerate cases</td>
</tr>
<tr>
<td><strong>Level 2</strong></td>
</tr>
<tr>
<td>We consider the logical structure of the problem, the number of deductive inferences in each resolution strategy, the possible approaches to solve the problem</td>
</tr>
</tbody>
</table>

Table 3.2: concepts and procedures for the resolution of the root-problem

In the following section we consider the cognitive messages established a priori. This is defined as the pedagogical space of the problem (Cobo et al., 2007). We do not consider the messages given ‘on the spot’ that are not designed a priori.

c) Pedagogic space of the problem

As mentioned in the previous section (3. Tutor orchestration), these messages are classified according to three levels. We also consider different messages for each phase of the resolution process. The students have to solve the root-problem with paper-and-pencil, thus we do not consider contextual messages. In the following table we condense the cognitive messages established for the different resolution phases and the different resolution strategies.

<table>
<thead>
<tr>
<th>Cognitive messages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Familiarization</strong></td>
</tr>
<tr>
<td>Level 1</td>
</tr>
<tr>
<td>Tracta de comprendre bé les condicions del problema</td>
</tr>
<tr>
<td>Identifica l’objectiu del problema</td>
</tr>
<tr>
<td>Torna a llegir l’enunciat del problema</td>
</tr>
<tr>
<td>Level 2</td>
</tr>
<tr>
<td>A quin punt de la diagonal hauria d’estar el punt M segons l’enunciat?</td>
</tr>
<tr>
<td>Recorda les propietats de les diagonals d’un rectangle</td>
</tr>
<tr>
<td>Què aporta el fet que les rectes per E siguin paral·leles als costats?</td>
</tr>
<tr>
<td>Level 3</td>
</tr>
<tr>
<td>Per a resoldre el primer apartat el raonament que has de fer és</td>
</tr>
</tbody>
</table>
Let P be any point on the median [AM] of a triangle ABC. Let m and n parallel lines through P to the sides (AB) and (AC) of the triangle.

a) What relation is there between the segments EM and MF?
b) Where must the point P be positioned such that BE = EF = FC?
The statement of this problem is more complex than the previous one (root problem). The concepts identified in the statement of the problem are simple: triangle, median of a triangle, parallelism, equality of segments. The difficulty of the statement can be justified in terms of the logical structure of the problem. The problem has two questions a) and b), and for both questions there are two nested quantified boxes. For the first question there are two quantifiers, while the property to find is in the third nested box (Tables 3.4 and 3.5). The students have to understand the logical dependence of the problem’s elements. We have considered this fact for assigning a level of difficulty to the problem. We consider also other aspects such as the number of resolution approaches identified in the basic space and the number of deductive steps for each resolution strategies. It is well known that a “pleasant-looking problem” does not necessarily have a “pleasant-solution” (Tao, 2006).

In this case we do not give the concrete lengths of the sides of the triangle, but we insert the triangle into a grid to facilitate students’ understanding of the problem. We insert the grid to help the students to visualize that the trisection of the segment BM is equivalent to the trisection of the segment BC. Nevertheless, we will observe in the analysis phase that the insertion of the triangle into a grid obstructs students’ understanding of the verbal statement of the problem. Also we considered introducing the first question as a hint for the second question. In the second question, the existential quantifier (Table 3.5) has a constructive flavour, for the verbal statement of the problem asks the students ‘to find’ a point P on the median such that BE=EF=FC. We expect the students to justify deductively the construction of this point, otherwise, they may find a robust construction without understanding the geometric properties of the construction. In this problem, we expect that the use of GeoGebra may help the students not only to conjecture the position of the point P on the median but also to understand the logical structure of the problem. Also it may allow the students to explore geometric properties of the figure leading to identify and justify deductively the position of the point P. Nevertheless, as Hoyles and Healy (1999) state, exploration of geometrical concepts using DGS helps the students to define and identify properties, but does not necessarily lead to the construction of a proof.

\[
\forall ABC \text{ triangle}
\]

\[
\begin{align*}
M & \text{ midpoint of the segment BC, } \forall P \in AM \\
\text{Parallel lines through P to the sides AC} & \quad \text{and AB that intersect BC in E, F respectively} \\
\rightarrow & \quad \Phi(EM, MF)
\end{align*}
\]

*Table 3.4: Logical structure for the first question of the scaled triangles problem*
∀ABC triangle

M midpoint of the segment BC, ∃P ∈ AM

Parallel lines through P to the sides AC
and AB that intersect BC in E, F respectively ∧ BE=EF=FC

Table 3.5: Logical structure for the second question of the scaled triangles problem

We summarize in the following paragraph the expectations concerning this second micro-cycle (problem of the scaled triangles).

Expectations concerning the second teaching experiment: problem of the scaled triangles (problem N3_2 GGB(III))

- Visualization of the algebraic relationships between quantities as segments. For example, the trisection of the segment BC is based on considering \( \frac{1}{3} \) of the segment BM which is \( \frac{1}{2} \) of the segment BC and thus \( \frac{1}{6} \) of BC. We obtain then \( \frac{2}{6} \) of the segment BC which is a trisection of the segment. We include the figure in a grid, such that BC=6u, to facilitate the GeoGebra construction and the visualization of this property. Moreover, in the first question we ask the students to show that EM=MF to suggest the algebraic relationship.

- Visualization to apply Thales property to other configurations than the usual one (higher order of steps and different configurations).

- The use of GeoGebra may act as a visual amplifier (Pea, 1995) in the task of identifying properties: Thales configurations, equal angles, similar triangles, the notion of centroid and its properties, etc.

- Understanding the concept of invariant (for example, M midpoint of EF, for any point P of the median) and thus understanding the need for a variety of constructions rather than just one. All dependent relationships are preserved through dragging, but all the elements have to be dragged according to the structure of the problem (quantifiers). Understanding the correspondence free/dependent GeoGebra object with logical relations of the problem (∀T triangle ABC, ∀P of the median AM, [M midpoint of EF]).

- Move from the spatial-graphical field to the theoretical field. GeoGebra as a tool to support the proving process (structuring, exploring, conjecturing, refuting). For example, in this problem the use of measures may be a tool for conjecturing and proving.
- Understanding the motion dependency in terms of logical dependency within the geometrical context, which is also a characteristic of the mathematical situation.
- The use of GeoGebra may foster explanatory proofs (on the other hand, the use of coordinates may hide its explanatory properties).

**a) Basic space of the scaled triangles problem**
In this section we describe different approaches to solve this problem. We have identified four resolution approaches. The first resolution approach is based on using Thales theorem to trisect a segment. There are different resolution strategies that correspond to this approach. The second resolution approach is based on geometric transformations (identifying homothetic triangles). We consider as the third approach, the analytic geometry approach. As the fourth approach of the basic space, we consider a resolution strategy based on particularization which can be generalized to the initial problem.

1. **Resolution approach: applying Thales theorem**
There are many approaches for this strategy. All these approaches are based on applying Thales theorem to trisect a segment for solving the second question. The first question can be solved using Thales theorem twice. The purpose of this question is to help the students to visualize that the trisection of the segment BM is equivalent to the trisection of the segment BC. We also include the figure in a grid with this purpose. We now present the different resolution strategies.

Resolution strategy for the first question based on applying Thales theorem:
We apply Thales theorem in two steps considering the similar triangles AMB, PME and the similar triangles AMC and PMF (Figure 3.3).

We obtain the equality of ratios: \[ \frac{MA}{MP} = \frac{MB}{ME} \quad \text{and} \quad \frac{MC}{MF} = \frac{MA}{MP} \Rightarrow \frac{MC}{MF} = \frac{MB}{ME} \]
As M is the midpoint of BC, we deduce from the previous equality that ME=MF.

- Resolution strategy of the second question based on the trisection of the segment AB:

![Figure 3.3: Thales theorem in the triangles MAC and MBA](image)

This strategy is based on the trisection of the side AB (Figure 3.4). We consider the parallel line, r, to the line (BC) through the point R (point of trisection of the segment AB). The parallel line to the line (AB) through the constructed point P intersects the side BC at the point E. Applying Thales theorem we obtain the
following equality of ratios: \( \frac{3}{1} = \frac{BA}{BR} = \frac{MA}{MP} = \frac{MB}{ME} \). Thus the point \( P \) constructed as the intersection point of the auxiliary line \( r \) and the median \( AM \) verifies the property.

*Figure 3.4: Trisection of the segment \( AB \) and parallel line to \( BC \) through \( R \)*

The resolution of this problem with GeoGebra, following this approach, requires the instrumented technique ‘trisection of a segment’ (Figure 3.4), which is based on the awareness of Thales theorem (mental part of the instrumented scheme).

- **One-step trisection of the segment \( BM \)**

For the second question, considering the problem solved, we obtain the equivalent problem of trisecting the segment \( BM \):

\[
BE = EF = FC \quad \Rightarrow \quad 3MF = MC \quad \text{(question a) \ EM=MF)}
\]

Applying Thales theorem we obtain \( MA=3MP \).

Analogously, we obtain the equivalent relation for the triangle \( BAM \). Using the previous relation, \( 3MF = MC \), we prove that \( E \) and \( F \) trisect the segment \( BC \).

\[
CF = \frac{2}{3} CM \quad \Rightarrow \quad 2CM = BC \quad \Rightarrow \quad CF = \frac{1}{3} BC
\]

The use of GeoGebra may help the students to interpret the relation between proportional combining the tool dragging and the tool distance between two points or length of a segment.

- **An equivalent strategy is based on the use of vectors and applying Thales theorem.** With GeoGebra we can introduce vectors and visualize their relations in the geometric window and in the algebraic window (Figure 3.5).
Figure 3.5: Vectorial resolution based on applying Thales theorem

\[ MP = kMA \iff MF = MP + PF = kMA + kAC = kMC \]

We apply Thales theorem for the case \( k = \frac{1}{3} \)

- Auxiliary element: the centroid of the triangle

This approach consists in considering two medians of the triangle (Figure 3.6). Applying the properties of the centroid, namely that the two medians trisect each other at the centroid \( P \) (which could have been defined as a point of trisection of one median) and applying Thales theorem, we obtain \( \frac{BM_1}{PM_1} = \frac{BC}{FC} \Rightarrow 3FC = BC \)

Figure 3.6: Centroid of the triangle ABC

To follow this approach, the students should know the properties of the centroid. The use of GeoGebra may help the students to visualize the position of the point \( P \) in the triangle \( ABC \). In this case the GeoGebra construction requires only elementary tools, but the deductive justification is not necessarily obtained from the experimentation with GeoGebra.
2. **Resolution strategy: Homothetic triangles**

This approach is based on the identification of similar triangles. In fact, these triangles, ABC and PEF, are in a homothetic configuration. We can apply the similarity criteria AAA (\(<P=<A, <B=<E, <F=<C\)). If BE=EF=FC, the base EF of the triangle EFP is a third of the base BC of the triangle ABC. Thus the ratio of similarity is one third and the homolog medians AM and PM verify also the relation AM= 3PM. This resolution is equivalent to considering the dilatation of centre M and ratio one third, \(H(M, \frac{1}{3})\), which transforms the triangle ABC into the triangle PEF such that:

\[H(A)=P, \ H(B)=E, \ H(C)=F\] (Figure 3.7).

![Figure 3.7: Dilatation of the triangle ABC with factor one third from the centre M](image)

The use of GeoGebra to follow this approach requires identifying similar triangles. We can construct the point P using the tool dilatation of an object from a point with a factor k. We can use a slider to introduce the factor k, and dragging the point P along the median to obtain BE=EF=FC or trisect the segment BC (applying Thales theorem). Due to round-off error, it is better to construct a triangle EFP and then obtain the triangle BAC with a dilatation of centre M and factor three.

3. **Resolution approach: Analytic geometry (based on finding the coordinates of the point E that trisects the segment and then we obtain the coordinates of P)**

This approach based on analytic geometry, requires finding appropriate coordinate axes in relation to the triangle. In this case, the tool polygon combined with the tool dragging objects may help the students to find appropriate coordinate axes in relation to the triangle. To follow this strategy, the students have to work out the distance between two points, find the equation of a straight line and solve linear systems of two equations and two unknowns.

4. **Resolution approach: particularization**

4.1 The case of an isosceles triangle: a strategy based on comparing areas

In the particular case of an isosceles triangle ABC in A, the median AM is also the height of the triangle from the vertex A. We can then consider strategies based on
equicomplementary dissection rules of areas. We obtain the relation between the heights of both triangles ABC and PEF and thus the relation between the medians.

a) In the particular case in which ABC is an isosceles triangle, we obtain the following equalities:

\[
\begin{align*}
(PME) &= k^2(ABM) \\
(PMF) &= k^2(AMC)
\end{align*}
\]

The median AM splits the triangle ABC in two triangles which have the same area, ABM and AMC ⇒ (PEM) = (PMF) ⇒ EM = MF (the triangles have the same area and heights with the same length).

b) The height of the isosceles triangle ABC from the vertex is also the median. If BE = EF = FC, then the area of the triangle ABC is 9 times the area of the triangle EPF (we apply Thales theorem). As the base EF is one third of the base BC, the height PM is one third of the height AM.

Generalization, for the case of an isosceles triangle:

We can generalize this particular case (isosceles triangle in A) by considering the dynamic strategy based on dragging the vertex A along a parallel line through A to the line (BC). Activating the trace of the point P and linking the point A to the parallel line, we obtain the trace of the point P (Figure 3.8a and Figure 3.8b).

![Figure 3.8 a: ABC isosceles triangle](image)

![Figure 3.8 b: Dragging A along the parallel we observe that the points E and F are invariants.](image)

Dragging A along the parallel line to (BC) (Figure 3.8b), the point P (dependent object) varies along the parallel line to (BC) through the initial point P. The ratio between the segments AP and PM is conserved and thus the points E and F are invariants. We deduce that P trisects the median. The use of tool dragging combined with the trace of the point P may also suggest the first strategy based on introducing the parallel line through the point T of the segment AB to the line (BC) such that 3BT = BA
4.2 The case of a right-angled triangle

We construct with GeoGebra a right-angled triangle ABC inscribed in a circumference of diameter BC (Figure 3.9). On dragging A along the circumference of radius AM and centre M (activating the trace of the point P), we observe that the point P is indirectly dragged along a circumference of centre M and radius ME. This procedure may suggest the resolution strategy based on considering dilatations (homothetic triangles).

![Figure 3.9: geometric locus](image)

**b) Table of contents, procedures and techniques (GGB problems)**

We condense in the following table (Table 3.6) the necessary conceptual knowledge, procedures and GeoGebra techniques to follow the different resolution strategies of the problem’s basic space.

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Mathematical content</th>
</tr>
</thead>
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<td>- triangle, median, parallelism, centroid...</td>
</tr>
<tr>
<td>figures (triangle, median,</td>
<td></td>
</tr>
<tr>
<td>parallelism, centroid...)</td>
<td></td>
</tr>
<tr>
<td>Triangles congruence criteria</td>
<td></td>
</tr>
<tr>
<td>Formula for the area of a</td>
<td></td>
</tr>
<tr>
<td>triangle</td>
<td></td>
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<tr>
<td>Relation between the angles</td>
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<tr>
<td>determined on parallel lines</td>
<td></td>
</tr>
<tr>
<td>by a secant line</td>
<td></td>
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<tr>
<td>Decomposition of areas</td>
<td></td>
</tr>
<tr>
<td>Thales theorem</td>
<td></td>
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<tr>
<td>Similarity of triangles</td>
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<tr>
<td>Triangles similarity criteria</td>
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<td>Relation between the areas</td>
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<td>of similar triangles and</td>
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<tr>
<td>the squared ratio of similarity</td>
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<tr>
<td>Dilatations and its properties</td>
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<tr>
<td>Coordinate axes</td>
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<td>Straight lines equations</td>
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<tr>
<td>Vectors and operations with</td>
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<tr>
<td>vectors</td>
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</table>

<table>
<thead>
<tr>
<th>Technical contents - Use of GeoGebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic construction tools (new point, midpoint, segment, polygon,...)</td>
</tr>
<tr>
<td>Parallel line to a given line through a point</td>
</tr>
<tr>
<td>Usage schemes of the dragging tool (syntax, order)</td>
</tr>
<tr>
<td>Definition of vectors in the input field and operations with vectors</td>
</tr>
<tr>
<td>Measure tools (distance, angles, areas...)</td>
</tr>
<tr>
<td>Activation of tools (hide/show object, properties, trace...)</td>
</tr>
<tr>
<td>Usage schemes for the algebraic and geometric window and for the input field</td>
</tr>
</tbody>
</table>
### Procedures and techniques

1. Applying Thales theorem (A, C)
2. Applying formula for the area of a triangle (B)
3. Applying congruence criteria of triangles (G)
4. Identifying and representing the heights and the medians of a triangle (B)
5. Applying equivalence of areas through complementary dissection rules (D, B)
6. Choosing coordinate axes (E)
7. Considering particular cases (F)
8. Determining similar triangles (H, G)
9. Identifying elements of the triangle such as the centroid (B)

A. Trisection of a segment based on Thales theorem (requires defining auxiliary elements)
B. Construction of the elements of a triangle (heights, medians, centroid) combined with the use of the tools distance and area
C. Combination of the tool dragging and the tool distance to discern in the algebraic window or geometric window proportional segments
D. Combination of the tool dragging and the tool area of a polygon to discern in the algebraic or geometric window relations between the areas of polygons
E. Dragging the vertices of the polygon or all the polygon with the tool axes activated
F. Using the dragging tool to obtain particular cases or to generalize particular solutions (tool dragging combined with the option trace of an object) to observe invariants
G. Applying measure tools to determine congruent and similar triangles
H. Using the dilatation tool combined with the tool slider to define as a parameter the similitude ratio

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Logical structure of the problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basic space of the problem</td>
</tr>
<tr>
<td></td>
<td>Inferential steps in each resolution approach</td>
</tr>
</tbody>
</table>

**Table 3.6**: contents for the resolution of the scaled triangles problem

c) Pedagogic space of the scaled triangles problem

<table>
<thead>
<tr>
<th>Cognitive messages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Familiarization</strong></td>
</tr>
</tbody>
</table>
| **Level 1** | Tracta de comprendre bé les condicions del problema  
Identifica l’objectiu del problema  
Torna a llegir l’enunciat del problema |
| **Level 2** | Reflexiona sobre a quins punts de la mitjana hauria de ser el punt P en cada apartat.  
Què aporta el fet de que AM sigui la mitjana?  
Com són els segments determinats sobre dues rectes tallades per paral·leles? |
| **Level 3** | Per a resoldre el primer apartat el raonament que has de fer és |
Table 3.7: Cognitive messages for the scaled triangles problem

<table>
<thead>
<tr>
<th>Contextual messages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Familiarization\Construction</strong></td>
</tr>
<tr>
<td><strong>Level 1</strong></td>
</tr>
<tr>
<td></td>
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<tr>
<td><strong>Level 2</strong></td>
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<td></td>
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<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Level 3</strong></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

**Analysis\ Exploration\ Planning\ Execution**

| **Level 1** | Ajuda sobre la sintaxis d’una eina determinada (eina distància, àrea, angle...) |
| **Level 2** | Busca relacions entre els elements de la figura desplaçant alguns elements  |
| | Tracta d’identificar propietats del baricentre |
| | Tracta d’identificar triangles semblants |
| **Level 3** | Tracta de considerar el costat comú dels dos triangles. Quina relació hi ha entre els segments en desplaçar P? |
| | Quina relació hi ha entre els segments determinats pel baricentre sobre la mitjana? |
| | Tracta de determinar triangles semblants |

**Verification**

| **Level 1** | Mira de comprovar el resultat que has obtingut en altres triangles  |
| **Level 2** | Pensa que has pogut aplicar alguna propietat de forma errònia. Revisa la construcció feta.  |
| **Level 3** | Pensa en altres formes de resoldre el problema  |

Table 3.8: Contextual messages for the scaled triangles problem
3.1.2.3 The median problem (GeoGebra and paper-and-pencil)

This is the third problem that the students are expected to solve in a technological environment (GeoGebra and paper-and-pencil). The logical structure of the problem is the same as that in the first question of the previous problem (Table 3.9). The concepts identified in the statement of the problem are: triangle, median and areas of triangles. The students have to find and justify deductively the relation between the areas of both inner triangles. In this problem the relation between the areas is simple to conjecture and the use of GeoGebra gives immediately this relation. Nevertheless, we present the statement as a ‘find’ question to maintain a similar structure for all the verbal statements of the problems. We also assign to this problem the third level of difficulty. As mentioned before, there is a progression in the way of introducing the problems and the statement of this problem is verbal, as we can see in the table below:

Let P be any point on the median [AM] of a triangle ABC. What relation is there between the areas of the triangles APB and APC?

<table>
<thead>
<tr>
<th>∀ABC triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>M midpoint of the segment BC, ∀P ∈ AM</td>
</tr>
<tr>
<td>P ∈ AM → Φ ((APB), (APC))</td>
</tr>
</tbody>
</table>

Table 3.9: logical structure of the median problem (third problem)

We summarize in the following paragraph the expectations concerning this third micro-cycle (problem of the scaled triangles).

Expectations concerning the third teaching experiment N3_3 GGB(III)
- Visualization of the elements of a triangle, such as exterior heights, discerning inner triangles
- Using GeoGebra as a visual amplifier (Pea, 1995) in the task of identifying properties: the notion of area and its properties, properties of the median, congruence of triangles
- Understanding the properties of areas as a measure function (theory of measures) in this particular context
- Understanding the concept of invariant (for example, the height from B of the triangle BAP is invariant while dragging P along the median AM), and thus understanding the need for a variety of constructions rather than just one.
- Moving from the spatial-graphical field to the theoretical field. GeoGebra as a tool to support the proving process (exploring, conjecturing, refuting).
- Understanding the motion dependency in terms of logical dependency within the geometrical context, which is also a characteristic of the mathematical situation.

**a) Basic space of the median problem**

We have identified five resolution approaches based on considering equality of ratios to compare the base and the heights of the inner triangles, decomposing the triangle in inner triangles of same area, considering an equivalent problem and considering a particular case (extreme position of the point P on the median, P=M).

1. **Decomposition of triangle into triangles that have the same area**

   This strategy is based on considering the inner ABP, BPC, APM and PMC. As the median of a triangle splits it into triangles of same area, we get the following equalities of areas:
   
   
   \[(ABM) = (BMC) \text{ and } (APM) = (PMC)\]
   
   By applying complementary dissection rules of areas, we get the equalities:
   
   \[(ABP) = (ABM)-(APM) = (BMC)-(PMC) = (BPC)\]

   ![Figure 3.10: Four inner triangles](image)

   The use of GeoGebra may help the students to observe that the inner triangles APM and PMC have also the same area. By using the tool polygon and dragging the point P along the median and the initial vertices of the triangle, we can observe in the algebraic window the equality of areas. This may suggest a strategy based on applying complementary dissection rules of areas.

2. **Auxiliary parallel line**

   This strategy is based on considering the parallel line through P to the line (AC). We obtain that the line (BM) is also a median of the triangle BDE (Figure 3.11). This can be proved applying Thales theorem. The point P is the midpoint of the segment DE (Figure 3.11).

   As the median of a triangle splits the triangle into two triangles of same area, we obtain the following equality of areas: \((BPD) = (BPE)\).

   The bases DP and PE of the triangles DPA and PEC have the same length, and the corresponding heights have also the same length (parallel segments comprised between parallel lines), thus the triangles have the same area.
Thus, the triangles APB and PBC have the same area:

\[(\text{APB}) = (\text{ADP}) + (\text{DPB}) = (\text{PEC}) + (\text{PEB}) = (\text{BPC})\]

The use of GeoGebra can help the students to consider this strategy, combining the tool dragging, the tool distance between two points and area of a polygon. If we construct the polygon DPA, GeoGebra labels the point A with a second letter A'. Dragging A' along the side AC we can observe that the area is invariant. We obtain an equivalent problem (Figure 3.12). For the particular position, P midpoint of the median, we obtain a parallelogram (Figure 3.12).

3. **Comparing the common base and the respective heights of the triangle APB and the triangle BPC**

This approach consists in proving that the heights FC and AE (Figure 3.13) have the same length. By applying Thales theorem or criteria of congruence (ASA) of triangles, we obtain the equality of heights. The right-angled triangles AME and MFC are congruent. To justify the congruence of triangles, we use the fact that the interior angles defined by secant line on two parallel lines are equal.
Figure 3.13: The heights AE and CF have the same measure

3. Equivalent problem: the median splits the triangle into triangles with the same area

Figure 3.14: Parallel lines e and f to the median CM and parallel line through C to the line (AB)

We modify the initial configuration, defining with the tool polygon the triangles CPD and CPE (Figure 3.14), which have the same area as the initial inner triangles, respectively (APC and CPB). As the triangles APC and PCD share the base CP and parallel segments comprised between parallel lines have the same length, the triangles have the same area. Considering the new configuration (Figure 3.14), we can apply the properties of the median to justify that both triangles have the same area. The use of GeoGebra can help to find equivalent problems through reconfigurable processes (dragging tool combined with measure tools and construction of auxiliary elements).
4. **Particularization**
Considering particular cases can be useful to find the resolution technique for the general case. The use of the tool dragging may help the students to find geometric properties of the figure. They can consider a larger family of triangles and observe continuous variations when dragging P along the median.

4.1 The degenerate case P=M: the triangle APC collapse in the segment AC
By dragging the point P along the median AM, we consider the degenerate case P=M, and we can observe that both triangles AMC and AMB have the same area. Using the properties of the median, we prove that these triangles have the same area.

We obtain the following general solution for any point P of the median:
1. Using the formula of area of a triangle, we deduce from the degenerate case that d(A, BM)=d(C, BM) (Figure 3.14). These triangles, ABM and BCM share the base BM and have the same area. Thus the corresponding heights have the same length.
2. Dragging P along median, the exterior heights of the triangles ACP and CPB from the vertices A and B, respectively, remain invariant (d(A, PB)=d(C, PB)). Thus these triangles have the same area (Figure 3.14).

c) **Table of contents, procedures and GeoGebra techniques**

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<tr>
<th>Mathematical content</th>
<th>Technical contents - Use of GeoGebra</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Definitions and elements of figures (triangle, median, parallelism ...)</td>
<td>Basic construction tools (new point, midpoint, segment, polygon, intersection of objects...)</td>
</tr>
<tr>
<td>Congruence of triangles criteria</td>
<td>Tool parallel line to a given line from a point</td>
</tr>
<tr>
<td>Formula for the area of a triangle</td>
<td>Tool perpendicular line to a given line from a point</td>
</tr>
<tr>
<td>Congruence of inner angles formed by a secant to parallel lines</td>
<td>Tool dragging (syntax)</td>
</tr>
<tr>
<td>Congruence of parallel segments comprised between parallel</td>
<td>Tools of measure (such as distance, angles, areas, etc.)</td>
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<tr>
<td>Decomposition of areas</td>
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<tr>
<td>The median splits the triangle in two triangles that have the same area</td>
<td>Tool trace</td>
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<tr>
<td>Thales’ theorem</td>
<td>Algebraic window, geometric window and input field</td>
</tr>
<tr>
<td>Identification of triangles in Thales configuration</td>
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</tr>
<tr>
<td>Concept of height as distance from a point to a line</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Procedures and techniques</th>
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</thead>
<tbody>
<tr>
<td>1. Applying Thales theorem (C)</td>
</tr>
<tr>
<td>2. Applying area formulas (B)</td>
</tr>
<tr>
<td>3. Applying triangles congruence criteria (G)</td>
</tr>
<tr>
<td>4. Identifying and representing the heights of a triangle (B)</td>
</tr>
<tr>
<td>5. Applying equality of areas for complementary dissection rules (D, B)</td>
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<td>8.</td>
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<td>9.</td>
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</tbody>
</table>

**Table 3.10:** mathematical content of the median problem (problem N3-3 GGB)

d) Pedagogic space of the problem of the median

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Logical structure of the problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basic space of the problem</td>
</tr>
<tr>
<td></td>
<td>Inferential steps in each resolution approach</td>
</tr>
</tbody>
</table>

**Cognitive messages**

<table>
<thead>
<tr>
<th>Familiarization</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level 1</strong></td>
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<table>
<thead>
<tr>
<th>Level 2</th>
<th>Què aporta el fet de que AM sigui la mitjana?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Com són els segments determinats sobre dues rectes tallades per paral·leles?</td>
</tr>
<tr>
<td></td>
<td>Recorda les propietats de les mitjanes d’un triangle</td>
</tr>
<tr>
<td></td>
<td>Quantes altures té un triangle?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Per a resoldre el primer apartat el raonament que has de fer és independent de la posició on es trobi el punt P</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>La mitjana d’un triangle el divideix en dos triangles de mateixa àrea</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Analysis\ Exploration\ Planning\ Execution</th>
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<tr>
<td><strong>Level 1</strong></td>
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<td></td>
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</table>

<table>
<thead>
<tr>
<th>Level 2</th>
<th>Intenta descompondre el triangle en d’altres triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tracta d’identificar triangles de mateixa àrea</td>
</tr>
<tr>
<td></td>
<td>Tracta d’identificar triangles congruents. Podries aplicar algun criteri de congruència?</td>
</tr>
<tr>
<td></td>
<td>Tracta de trobar relacions entre les altures les altures dels triangles, les àrees dels triangles que vols comparar</td>
</tr>
<tr>
<td></td>
<td>Quina altura del triangle et convé? Quina base et convé?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Podries descompondre el triangle en quatre triangles de vèrtex P</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Intenta resoldre el problema que resulta de situar el punt P a la posició del vèrtex A</td>
</tr>
<tr>
<td></td>
<td>Pots agafar com a bases del triangle el costat comú AP. En aquest cas construeix les altures</td>
</tr>
</tbody>
</table>
Consider the case of an isosceles triangle in $A$, is the procedure possible generalizing to an isosceles triangle? How would you justify that the triangles are congruent?

<table>
<thead>
<tr>
<th>Verification</th>
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<tbody>
<tr>
<td>Level 1</td>
</tr>
<tr>
<td>Level 2</td>
</tr>
<tr>
<td>Level 3</td>
</tr>
</tbody>
</table>

Table 3.11: Cognitive messages for the median problem

<table>
<thead>
<tr>
<th>Contextual messages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Familiarization/Construction</strong></td>
</tr>
<tr>
<td>Level 1</td>
</tr>
<tr>
<td>Level 2</td>
</tr>
<tr>
<td>Level 3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Analysis\ Exploration\ Planning\ Execution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
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<tr>
<td>Level 2</td>
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<tr>
<td>Level 3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Verification</th>
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</thead>
<tbody>
<tr>
<td>Level 1</td>
</tr>
<tr>
<td>Level 2</td>
</tr>
<tr>
<td>Level 3</td>
</tr>
</tbody>
</table>

Table 3.12: Contextual messages for the median problem

### 3.1.2.4 The quadrilateral problem (paper-and-pencil)

This problem is the fourth problem the students are set and has to be solved only with paper-and-pencil. The statement and the logical structure of the problem are similar to the previous problem. The difference with respect to the previous problem is that we introduce the area of a quadrilateral into the statement. Moreover, the students have to compare this area with the addition of areas of two inner triangles, and both areas are constant when varying the point $P$. We also assign the third level of difficulty.
Let ABC a triangle and let P any point of the side BC, N and M be the midpoints of the sides AB and AC respectively.

What relation is there between the area of the quadrilateral ANPM and the addition of the areas of the triangles BNP and PMC?

The logical structure of the problem is the same as in the first question of the previous problem (Table 3.13)

∀ABC triangle

N midpoint of AB, M midpoint of AC, ∀P ∈ BC

P ∈ BC → Φ((ANPM), (BNP) + (PMC))

Table 3.13: Logical structure of the quadrilateral problem

We summarize in the following paragraph the expectations concerning this fourth microcycle (problem of the quadrilateral).

**Expectations concerning the fourth teaching experiment: the quadrilateral problem (N3_4 Paper-and-pencil (III))**

- Understanding the logical structure of the problem
- Understanding triangulation properties to obtain the area of the figure
- Visualizing the dynamic properties of the figure
- Understanding and applying the reciprocal of Thales theorem
- Understanding the invariant properties of the figure (the areas to compare are constant when varying P along the base of the triangle)
- Understanding equidecomposition by dissection
- Understanding the properties of the area function in this context
- Understanding the relation between similar triangles and ratio of areas

**a) Basic space of the quadrilateral problem**

In the following paragraph, we describe the main resolution approaches that form the basic space of the problem. We distinguish four main approaches that are based on: decomposition and comparison of areas; obtaining equality of ratios to compare the base and the height of the inner triangles, and strategies based on varying slightly the problem (consideration of extreme and particular cases that are equivalent to the general case).

1. **Strategy based on decomposition and comparison of areas**
This strategy is based on introducing an auxiliary segment $AP$ that splits the outside triangle into four inner triangles. The median $PM$ splits the triangle $ABP$ (Figure 3.15) into two triangles that have the same area: $(BPM) = (PNA)$. Analogously, the median $PM$ splits the triangle $PAC$ into two triangles that have the same area: $(APM) = (PMC)$.

Thus the areas to compare are equal:

$$(ANPM) = (ANP) + (APM) = (BPN) + (PMC)$$

![Figure 3.15: Auxiliary segment AP that decomposes the figure in four inner triangles](image)

2. **Strategy based on obtaining equality of ratios to compare the base and height of the triangles**

We consider the segment $MN$ that splits the quadrilateral into two inner triangles, $ANM$ and $NPM$. By applying the reciprocal of Thales theorem, we deduce that the lines $MN$ and $BC$ are parallel. As parallel segments comprised between parallel lines are equal, the heights from $M$ and $N$ (Figure 3.16) of the triangles $BNP$ and $PMC$ have the same length. By applying Thales theorem, the height from the vertex $A$ of the triangle $ANM$ has also the same length, which is half the length of the height from the vertex $A$ of the triangle $ABC$. We obtain:

$$\frac{1}{2} (ABC) \Leftrightarrow \frac{1}{2} (ANPM) \Leftrightarrow (BNP)+(PMC) = (ANPM)$$
A variant of this strategy is based on comparing heights and bases and applying the area formula to obtain separately the areas of the quadrilateral and the addition of areas of both triangles. Through the comparison of both expressions of the areas, we obtain the equality of areas.

3. **Strategies based on considering degenerate and particular cases**

These strategies are based on the concept of height as distance from a point to a line. Through reconfigurative operative apprehension, we obtain the following equivalent problems:

The median splits the triangle into two triangles of same area. The addition of the area of the triangles NBP and PMC is equal to the area of the triangle BMC, which is half the area of the triangle ABC.

The addition of the areas of the triangles BNP and PMC is independent of the position of the point P on the side BC. In this case, the four inner triangles are congruent.

**b) Table of contents and procedures**

We condense in the following table the concepts and techniques used for the resolution of these problems according to the basic space of the problem.
<table>
<thead>
<tr>
<th>Concepts</th>
<th>Mathematical contents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Definitions and elements of the figure (triangle, quadrilateral, parallelism, etc.)</td>
</tr>
<tr>
<td></td>
<td>Congruent triangles</td>
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<tr>
<td></td>
<td>Criteria of congruence of triangles</td>
</tr>
<tr>
<td></td>
<td>Similar triangles</td>
</tr>
<tr>
<td></td>
<td>Criteria of similarity of triangles</td>
</tr>
<tr>
<td></td>
<td>Congruence of parallel segments comprised between parallel lines</td>
</tr>
<tr>
<td></td>
<td>Thales theorem</td>
</tr>
<tr>
<td></td>
<td>Reciprocal of Thales theorem</td>
</tr>
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<td></td>
<td>Formula for the area of a triangle</td>
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<tr>
<td></td>
<td>Ratio between homolog sides of similar triangles</td>
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<tr>
<td></td>
<td>Squared ratio of areas of similar triangles</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Procedures and techniques</th>
<th></th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Identifying triangles in Thales configuration</td>
</tr>
<tr>
<td></td>
<td>Applying Thales theorem</td>
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<tr>
<td></td>
<td>Applying congruence criteria of triangles</td>
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<tr>
<td></td>
<td>Applying similarity criteria of triangles</td>
</tr>
<tr>
<td></td>
<td>Identifying and representing the heights of a triangle</td>
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<tr>
<td></td>
<td>Applying the formula of the area of a triangle</td>
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<tr>
<td></td>
<td>Identifying congruent triangles</td>
</tr>
<tr>
<td></td>
<td>Identifying similar triangles</td>
</tr>
<tr>
<td></td>
<td>Applying the relation between the ratio of similarity to the areas</td>
</tr>
<tr>
<td></td>
<td>Equivalence of areas and complementary dissection rules</td>
</tr>
<tr>
<td></td>
<td>Decomposing a triangle and a quadrilateral in other triangles</td>
</tr>
<tr>
<td></td>
<td>Considering particular and extreme cases</td>
</tr>
</tbody>
</table>

| Level 3 | Logical structure of the problem |
|         | Basic space of the problem |
|         | Inferential steps in each resolution approach |

Table 3.14: Concepts and procedures

d) Pedagogic space of the problem

We condense in the following table the cognitive messages established a priori.

<table>
<thead>
<tr>
<th>Cognitive messages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Familiarization</strong></td>
</tr>
<tr>
<td>Level 1</td>
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<td>Level 3</td>
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<table>
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<th>Analysis\ Exploration\ Planning\ Execution</th>
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<tr>
<td>Level 3</td>
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</table>
Table 3.15: Cognitive messages

### 3.2 Methodology

We expect to answer the second research question following an observational methodology by analysing observable behaviours (interactions between agents and technical aspects of instrumented schemes). We follow a case study and analyse it from a qualitative perspective. We describe in the following sections the sample, the teaching experiment, the data collected, and finally we focus on the data analysis method of the study.

#### 3.2.1 The sample

The study is conducted with a group of twelve 17-year-old students selected from a regular class in a public high school in Catalonia, Spain. These students, who have been selected by their teacher, are high-achieving students who are interested in participating in the teaching experiment. We decided to select communicative students who were motivated in the experiment to facilitate the extraction of data. We decided to carry out the study in this particular high-school because these students are used to working on Euclidean geometry in problem solving dynamics, and their teacher takes the time needed to elicit students’ thinking rather than quickly giving them the answer. Moreover, their teacher is also a researcher in the group.

Another decision concerning the sample is to carry out the teaching experiment with students that are in the first term of the 12th-grade (segon de Batxillerat Tecnològic\(^9\) in the Catalan educational system). These students have the necessary knowledge to solve the proposed problems with different resolution strategies, and they are not studying geometry in mathematics class. The mathematical content of the problems was dealt with in courses prior to the one the students are currently taking. The students have procedural knowledge related to the problems proposed. The purpose of this decision is to prevent the concepts studied at the moment of the teaching experiment from having any influence on the students’ resolution strategies. For instance, the learning of trigonometry may have influenced their solving strategies in the root problem. As we are interested in the analyzing students behaviours when solving problems influenced by three variables (instructional design, orchestration, synergy of environments), we try to avoid the influence of other variables, such as the mathematical topic studied at the moment. Another relevant aspect of the sample is that these students are not used to working with

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\(^9\) www.gencat.cat/educacio
dynamic geometry software. They are novice users of GeoGebra. We take into account this fact to design the instructional sequence and thus we prepare two GeoGebra introductory sessions.

All twelve students participating in the teaching experiment are high-achieving students who are expressive and motivated in participating. Their teacher informs us about the most relevant students’ cognitive characteristics. Also we use the root problem as an initial test of students’ knowledge concerning the contents and procedures of the selected problems. From the analysis of protocols we select three prototypic profiles. We report in this study the in-depth analysis of the learning trajectories of three students who represent the profiles considered. For considering prototypic cases, we take into account the following aspects: the students’ cognitive orientation understood in the sense of Kruteskii’s (1976) categories (geometric, analytic and harmonic), the confidence of these students on their resolution strategies. We interpret these aspects in terms of profile characteristics. Nevertheless, a student can show different profiles in different tasks.

### 3.2.2 The teaching experiment

All the activities with the students were planned to include one previous two-hour introductory session of GeoGebra and two sessions of two hours each. The instructional sequence is planned for a teaching experiment lasting four hours carried out over a week in sessions of two hours. We do not consider a longitudinal study and try to concentrate the sessions over a week to avoid interference from other agents (for instance, the teaching of concepts that may interfere with the proposed problems) and also to enable the students to concentrate on the experiment and the use of GeoGebra.

We prepared the introductory sessions by considering the mathematical concepts and procedures required for the tasks of instructional design. We considered all the tools that were useful¹⁰ for the tasks proposed in the instructional design.

For the introductory session, we created a booklet to teach the students the useful tools for the tasks. We introduced GeoGebra with some examples in which the students had to use the basic objects in GeoGebra (points, segments, straight lines, polygons, circles, etc.) and the basic usage schemes they would need for the following sessions. We remarked special features of GeoGebra such as the relationship between the geometric aspects of figures and their algebraic representations and the dragging tool. As part of the didactical contract, we pointed out the necessity of constructing figures based on geometric properties to obtain constructions that are not ‘messed-up’ by dragging. Another important fact was to choose examples that did not interfere with the problems of the following sessions. For these reasons we designed the introductory booklet after designing the instructional sequence. This session was a whole-group session guided by their teacher and each student had his own computer. We do not analyse this introductory session.

During the teaching experiment the students worked individually in the computer-room. We considered the possibility of having the students work in pairs to obtain information

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¹⁰ From a geometric point of view, the tools new point, circle through a point given the centre and intersection of objects are sufficient to build all the others. We consider all the tools even if some of them are not necessary.
about their resolution processes more easily. Finally we rejected this option because introducing student-student interactions made it difficult to analyse the interactions between the student, the tutor, the environment and the content. For this reason, we have considered students who are expressive and are able to explain their resolution process while they are working on the problem. Moreover, the tutor encourages the students to express their actions with messages such as: *what are you doing?, why are you doing that?* In the following paragraph we describe the two sessions of the teaching experiment. In each session the twelve students worked in the computer-room.

1. First session (2 hours): the students have to solve individually the root problem with paper-and-pencil. Each student has his own worksheet with the statement of the problem (Figure 3.16). In the worksheet there are also some instructions about the resolution process: a) the tutor can give them some hints (cognitive and contextual messages) in the different phases of the resolution process, b) the students are supposed to explain their resolution process and c) the students should inform the tutor once they have finished the problem (Figure 3.16 and Figure 3.17). Taking into account the instructional design, the tutor assigns the next problem to each student considering the resolution process followed by the student. He also indicates to the student the environment in which each problem has to be solved (only paper-and-pencil or with GeoGebra and paper-and-pencil). During the session, the tutor observes and guides each student with cognitive and contextual messages. This is the second part of the orchestration. For each problem, the tutor has a document, the pedagogical space of the problem, with the cognitive and contextual messages. Nevertheless, some students’ difficulties are not considered in the a priori analysis of the problem and the tutor has to take ‘ad hoc’ decisions. We explain in detail the tutor’s orchestration in section 3. The tutor role. During the first session, each student works at ‘his own speed’ and we do not set the student a new problem if there is not enough time (less than 20 minutes). Some students finish earlier and other students need extra-time to finish the problem.

2. Second session (two hours): during the second session we follow the same process as in the first session. Each student starts with the assigned problem of the instructional design.

<table>
<thead>
<tr>
<th>Problema Llapis Paper</th>
<th>Nom i cognom:</th>
<th>Data:</th>
<th>Centre:</th>
</tr>
</thead>
</table>

**Indicacions:** El tutor et pot donar tres tipus d’ajudes.

- Ajudes per la comprensió de l’enunciat del problema
- Ajudes en la resolució del problema
- Ajudes sobre la comprovació de la teva solució

**RESOLUCIÓ**

És molt important que expliquis el que fas, perquè ho fas i com ho fas. Quan hagis acabat el problema o en cas de dubtes demana l’ajuda del tutor que et donarà les possibles indicacions o bé et
proposarà un altre problema.

**Figure 3.16:** Student’s worksheet for a paper-and-pencil problem

<table>
<thead>
<tr>
<th>Problema: GeoGebra</th>
<th>Nom i cognom:</th>
<th>Data:</th>
<th>Centre:</th>
</tr>
</thead>
</table>

**Indicacions:** El tutor et pot donar tres tipus d’ajudes.
- Ajudes per la comprensió de l’enunciat del problema i per l’ús de GeoGebra
- Ajudes en la resolució del problema
- Ajudes sobre la comprovació de la teva solució

**RESOLUCIÓ**
Es molt important que expliquis amb llapis i paper el que fas, perquè ho fas i com ho fas. Quan hagis acabat el problema o en cas de dubtes demana l’ajuda del tutor que et donarà les possibles indicacions o bé et proposarà un altre problema.

Desa l’arxiu de Geogebra amb el teu nom_probN3_2(III).ggb quan hagis acabat.

1. Intenta entendre l’enunciat i fes la construcció associada a l’enunciat amb GeoGebra. Recorda que has de justificar amb llapis i paper la resolució del problema.

**Figure 3.17:** Student’s worksheet for a GeoGebra problem

For the analysis phase, the whole set of data is: a) the written protocols, b) the audio and video-taped interactions within the classroom (student-tutor, student-environment), c) the GeoGebra files and the screen-records for each GGB problem and d) the itinerary followed by the student (local cycle) and tutor’s messages for each student. All this data is examined in order to inform about our research goals. In the following section we explain the way in which we carry out the analysis of data.

### 3.2.3 Analysis of a local cycle (third itinerary of problems)

A local cycle is an itinerary of problems. We report the analysis of the itinerary followed by the three prototypic students and we focus on the analysis of four consecutive problems from the third itinerary. We analyse the resolution process of each problem with data coming from the tapes, the written protocols and the screen-records. Taking into account the conceptual framework and the objectives of the research, we propose the following cyclic analysis for this itinerary. We analyse each problem as a micro-cycle and for each problem we reflect on the findings and we formulate the feed-forward. We describe firstly the analysis of a micro-cycle (problem) which has been carried out following four steps. Secondly we compare each micro-cycle considering different variables, such as the task, the tutor’s orchestration and the environment.

The objective of the analysis of each problem is to understand the effect it has on the student through the different geometrical competences, taking into account the expected effect of the problem at the start, the tutor’s orchestration and the environment in which the problem is solved. After the analysis of a micro-cycle, we describe the transitions for each competence between the consecutive problems, giving explanations for the differences between the acquisition of geometric competences (visual, structural,
instrumental and deductive). We characterize the learning trajectory of a student in terms of these transitions. For the analysis of each problem we follow four phases. In the following sections we detail these four phases.

### 3.2.3.1 Protocol partition

Firstly, we transcribe and divide the protocols into episodes, which are periods of time in which the student follows a phase of the resolution process (Schoenfeld, 1985) without transitions such as long silences or tutor’s interactions that make the students change their solving path, etc. We borrow this way of dividing the protocol from Puig (1996). To identify each episode, we adapt to our research the system adapted by Cobo (1998) for identifying the different phases of the resolution process. The most relevant change is that we introduce aspects to identify the different phases referring to the use of GeoGebra and the tutor’s orchestration. In the following paragraph, we mention the phases considered and the respective indicators of each phase.

#### Familiarization phase

This refers to the aspects that allow the students to understand the statement of the problem:
- Reading the statement
- Silences after reading the statement
- Writing on the worksheet the data of the problem and the objective
- Drawing a graphic representation or constructing a figure with GeoGebra to represent or understand the data of the problem
- Dragging elements of the figure to understand the problem’s statement
- Asking the tutor questions about the statement of the problem or about the way of constructing the associated figure with GeoGebra.

#### Analysis phase

In this phase the student introduces new information and his actions are structured and have a clear objective:
- Introduction of information in a structured way
- Search for resolution approaches to solving the problem, such as: introducing auxiliary elements, considering equivalent problems, choosing an adequate system of representation, expressing formulas, etc.
- Modifying the figure through dragging to understand the logical structure of the problem and to search a resolution approach.
- Search for relations between the data of the problem and the objective
- Selection of a resolution approach (this also can be due to the influence of the tutor, the environment, etc.)

#### Exploration phase

This is also a phase in which the student searches for new data about the problem. Nevertheless, this search is not structured and does not have a clear objective:
- The student tries to apply heuristic strategies
- The student drags the elements of the figure searching for geometric properties of the figure (wandering dragging)
- The student tries to identify information considering the conditions of the problem, the objective of the problem

**Planning phase**
- The student explains his resolution plan to the tutor and asks for validation
- The student plans the resolution strategy following the approach considered
- The student does not explain his plan and it is implicit in his resolution strategy
- The student drags elements of the figure or constructs auxiliary elements to carry out the resolution process he has planned

**Execution phase**
- The student applies the resolution process planned
- The student tries to obtain equalities of ratios and to apply formulas in order to compare the elements of the figure
- The student identifies relations and tries to justify them (argumentatively or deductively)
- The student constructs elements of the figure and modifies the figure to obtain other configurations

**Verification and local evaluation phases**
It consists in the evaluation of the resolution process. We include local evaluations of the episodes.
- Evaluation of the different episodes of the resolution process
- Validation of the results: the student asks the tutor for validation messages, the student checks with GeoGebra the results obtained (through perceptual apprehension, using measure tools, using the dragging tool, using the ‘relation between objects tool’, etc.)
- Evaluation of the results after a message from the tutor

**Transitions**
A transition indicates the resolution phase in which there is a relevant change in the resolution process. The student abandons the solving path due to: an evaluation of the solving process that makes the student change his solving path, an intervention of the tutor.

For each episode, we organize the protocol in a table. We enumerate the student’s dialogues and actions. We use a table to describe each episode in terms of dialogues and construction steps, strategy, argumentation and visualization processes. In the first column we indicate the dialogues (student-tutor) and the students’ actions (graphic actions, drawings, actions on the computer, verbal affirmations, etc.). In the second column we identify the strategy followed by the student (heuristic strategies and role) and we classify the messages given by the tutor and the messages’ role. In the third column we identify inferential steps considering the set of inferences of Duval (1995). We add the *figurative (or figural) inference* (Richard, 2004b) to the forms of discursive
reasoning. As stated by Richard (2004b), the figurative inference constitutes recognition of an existing bridge between the geometric figure and the process of proof. Within the cooperation between the role of the drawing and the role of the reasoning, if Presmeg (1986) deduces the importance of reasoning in the development of visual images, the figurative inference grants to the drawing an authentic role of justification in a step of reasoning. This is why, in adding this type of inference to the forms of discursive reasoning, we complete the set of inferences of Duval (1995) by a consistent extension that respects the foundation of the functional definition of reasoning.

Finally, in the last column, we identify the visualization processes. We also describe the modifications through dragging of the figure. For instance, for the problems to be solved with GeoGebra, we distinguish the different kinds of dragging defined in the theoretical framework (section 2.1 the instrumental approach) and its role in the solving process. We use these columns to relate the visualization and reasoning processes in solving geometry problems that require a proof (Torregrosa & Quesada, 2008). As an example, in the following table we consider an episode which corresponds to an execution phase.

<table>
<thead>
<tr>
<th>Dialogues and construction steps</th>
<th>Strategy</th>
<th>Argumentation</th>
<th>Figure (P&amp;P statement, GGB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23. Guillem: La base sempre serà la mateixa. Llavors, he de demostrar perquè les altures són iguals... [He remains silent observing the static figure].</td>
<td>He tries to justify(^1) the right-angled triangles BHM and MFC are congruent</td>
<td>Deductive reasoning</td>
<td>Perceptual apprehension: identification of the elements of the triangle on the static figure</td>
</tr>
<tr>
<td>24. Guillem: No sé... la hipotenusa d’aquests dos triangles és la mateixa [BHM and MFC] perqué M és el punt mig. El que faltaria és... [HM = MF] He tries to justify that the right-angled triangles BHM and MFC are congruent</td>
<td>He deduces that the hypotenuses of both triangles are equal BM=MC</td>
<td>Operative apprehension: he extracts both right-angled triangles from the initial configuration. He applies a change of anchorage from visual to discursive</td>
<td></td>
</tr>
<tr>
<td>25. Guillem: Ja està. tenen un angle igual i un costat igual... Undefined: we wonder if he considers the congruence criteria or applying trigonometry of the right-angled triangle.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26. Tutor: És suficient? Cognitive message of level 1 for the validation phase</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27. Guillem (35 min): Clar. Si tenen la hipotenusa i un angle el cosinus sempre serà igual... He applies trigonometry of the right-angled triangle.</td>
<td>Deductive reasoning</td>
<td>Discursive apprehension</td>
<td></td>
</tr>
</tbody>
</table>

\(^{11}\) We use the term justification to refer to any reason given to convince people (e.g., teachers and other students) of the truth of a statement (Marrades & Gutiérrez, 2000).
We consider the steps prior to carrying out the third step of the analysis, which consists in a microanalysis of each episode. In the following section we describe this third step.

### Table 3.19: Hypothetical learning trajectory (based on an ID that focuses on similarity theory an Euclidean area)

<table>
<thead>
<tr>
<th>Competence</th>
<th>Geometrical meaning</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visual</td>
<td>Geometric objects</td>
<td>Construction GGB (drawing)</td>
</tr>
<tr>
<td>Structural</td>
<td>Theorems and properties</td>
<td>Construction GGB (Figure)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>Necessity of proof</td>
<td>Protocol of construction</td>
</tr>
<tr>
<td>Deductive</td>
<td>Deductive proof</td>
<td>Figural inference</td>
</tr>
</tbody>
</table>

#### 3.2.3.3 Summary of the protocol: applying theory to practice

We then analyse the resolution process focusing, within each episode, in the following aspects:

**a) The cognitive aspect**

We identify the mathematical contents mentioned and used by the students, the mathematical contents used and not mentioned, the learning opportunities and the difficulties encountered. We identify fragments of protocol that give insight into the acquisition of geometric competences and fragments that give insight into conceptions concerning the competence that are not adequate. To analyse these fragments we use a coding system, defined in the theoretical framework, which is based on four levels defined with different scores.

**b) Influence of the tutor on the student**

We analyse whether the student requests the help of the tutor or not, whether they follow the suggestions of the tutor and the effect of these suggestions on the students. We
consider the different levels of messages that the tutor can give to analyse the influence of the tutor.

c) Instrumental aspect
We analyse the role of GeoGebra and the synergy and connectivity of both environments. We consider the different phases in which students use GeoGebra and we detect the schemes of usage and instrumented schemes (section 2.1 The instrumental approach).

The three aspects considered above are intertwined. For instance, some cognitive benefits stem from the interactions of the student with the tutor and from the synergy of environments. We analyse each episode taking into account these three aspects globally. For each episode, we analyse the interactions between the task, the student, the tutor and the environment and the effect of these interactions on student’s acquisition of geometric competences.
Finally, we summarize the scores referring to each competence and interpret the effect of the micro-cycle on the student (feed-forward concerning the local HLT).

3.2.3.4 Feed-forward concerning the local Hypothetical Learning Trajectory
For each problem, we condense in a table scores referring to the acquisition degrees of the geometrical competences. We adapt Drijvers’s (2003) coding system. The (++) scores refer to observations that reveal a deep insight into the competence; the (+) scores refer to observations that reveal the expected insight into the competence; the (o) scores refer to insights into the competence that are not completely correct but do contain valuable elements or observations done with the help of the tutor, and the (-) scores refer to conceptions concerning the competence that are not adequate. We show in the following tables the indicators for assigning the different scores to each competence. These indicators are assigned taking into account the problems set and the students participating in the teaching experiment. We define for each competence the meaning of each score and define indicators for these scores. Finally, we show some examples for each competence.

Visualization competence
We define the meaning of each score concerning the visualization competence, and we give some indicators for each score.

- Visualization (-) scores: The students try to visualize the problem but their visual skills are poor. This is reflected in the difficulty in visualizing elements even in standard configurations. Lack of internalization of the distinction between drawing and figure.
- Visualization (o) scores: We observe evidence of the ability of the students to visualize certain aspects (known results for standard configurations), but in non-standard configurations they have difficulties. For instance, they have difficulties in visualizing the exterior height of a triangle, triangles in non-standard Thales configuration, etc. Partial internalization of the distinction between drawing and figure.
- Visualization (+) scores: The students are able to apply visually theorems and definitions in non-standard configurations. They discern inner figures such as
similar and equivalent figures. They apply *kinaesthetic imagery* (or physical movement). When using GeoGebra, they drag the elements in a small area and it corresponds to *bound dragging* (dragging a point linked to a segment).

- Visualization (++) scores: The students have deep visualization skills. We observe unexpected original visualization processes for students of this age (12th grade) as for instance, reconfigurative visualization processes (*dynamic imagery*).

We define in the following table the indicators for the different visualization scores (table 3.20)

| Visualization competence | (-) scores | 1 Difficulty in discerning properties of the drawing from properties of the figure in standard cases (For instance equidistance, perpendicularity) |
| ~ | 2 The students are easily misled by the position of the figure. For instance, difficulties in visualizing of the height of a triangle in an oblique position |
| ~ | 3 Inability to think about the variation of a particular figure. A rigorous static (non-dynamic) conception of drawing. |
| (o) scores | 1 Visualization of properties in standard configurations whereas in non-standard configurations easily misled. For instance: exterior heights of a triangle, trisection of a segment (horizontal vs. oblique), Thales configuration |
| ~ | 2 Partial visualization of geometric properties (use of irrelevant visual properties) |
| ~ | 3 Operative apprehension: extraction of congruent figures but difficulty with equivalent, similar figures |
| (+) scores | 1 Visualization of standard geometric properties of the figure: clear distinction between drawing and figure |
| ~ | 2 Operative apprehension: extraction of similar figures, equivalent figures |
| ~ | 3 Operative apprehension: Introduction of auxiliary elements (we consider in the structural competence when is it is part of the elaboration of the problem) |
| ~ | 4 Dynamic visualization: think about the variation of a point in a reduced area or linked variation (for instance, dragging a point along a segment) |
| (+++) scores | 1 Reconfigurative visualization: for instance, transformation of two shapes in an equivalent shape. (it can be considered in the deductive competence for the case of visual proofs) |
| ~ | 2 Visualization algebraic-geometric: relationships of segments as quantities |
| ~ | 3 Extraction of equivalent figures and mental visual transformations in terms of areas, for instance. |

**Table 3.20: Visualization codes**

**Structural competence**

We define the meaning of each score concerning the structural competence, and we give some indicators for each score.
• Structural (-) scores: The student is not able to work out the different elements of the problem. He does not apply properties; he does not introduce auxiliary elements. The student has difficulties in discerning free objects from dependent objects.
• Structural (0) scores: Partial understanding of the elements of the problem. Straightforward application of well-known theorems in standard configurations. The students do not construct auxiliary elements. There is evidence of partial insight into the understanding of the statement of the problem: free/dependent objects.
• Structural (+) scores: The students apply well-known theorems in non-standard situations. Elaboration and construction of auxiliary elements. Understanding of the statement of the problem: free/dependent objects.
• Structural (++) scores: We observe unexpected elaborations of the problem, reformulations of the problem. Deep understanding of the statement of the problem: free/dependent object.

We define in the following table the indicators for the scores concerning the structural competence (table 3.21).

| Structural competence | (-) scores | | (o) scores | | (+) scores | | (+++) scores |
|-----------------------|------------|------------------|------------------|------------------|------------------|------------------|
|                       | 1. Lack of idea in discerning between free/dependent objects of the figure. For instance, distinction from arbitrary objects/concrete objects | | 1. Partial ability to apply theorems, identify definitions | | 1. Applying theorems and elaborating definitions | | 1. Identification, construction of auxiliary elements which are not standard (elaboration of the problem). For instance, applying |
|                       | 2. Lack of knowledge concerning the basic concepts that these students should have | | 2. Partial distinction between figure and drawing (difficulties in discerning geometric properties of the figure, use of particular properties of the drawing as hypothesis) | | 2. Distinction between figure and drawing | | |
|                       | 3. Inability to understand the statement of the problem. For instance, difficulty in identifying the basic objects of the statement | | 3. Partial understanding of the logical structure of the problem (difficulties) | | 3. Use of auxiliary standard elements to complete the figure. For instance extensive use of auxiliary objects proportioned by the software and correct apprehensions of the information given by the software. | | |
|                       | 4. Easily misled with standard configurations (simple logical structure for instance) | | 4. Partial understanding of the ontological status of objects (GGB). | | 4. Understanding of the logical structure of the problem | | |
|                       | 5. Wrong understanding of the quantifier nature. For instance, considering special cases of triangles as general case. | | | | 5. Understanding of the ontological status of objects (GGB) | | |
geometric transformations (dilatation), construction of figures that are not homolog (transformations that preserve area).

2. Connecting different resolution approaches (algebraic-geometric). For instance, combined use of algebraic and geometric window to obtain information.

3. Reformulating the problem in an equivalent problem

4. Awareness of the deep relation between the logical structure of the problem and the ontological status of the objects.

Table 3.21: Structural codes

**Instrumental competence**

We define the meaning of each score concerning the instrumental competence, and we give some indicators for each score.

- Instrumental (-) scores: Lack of necessity of proof. Lack of necessity of looking for solving strategies
- Instrumental (o) scores: The students understand the necessity of proving but they have a *naïve proof searching*. They do not consider heuristics such as particularization, contradiction, supposing the problem solved, etc. This partial or naïve proof search is related to a weak structural competence.
- Instrumental (+) scores: Necessity of proof and evidence of proof search. Evidence of heuristics of proof search, such as figural inference (visual evidence of proof search), elaboration and derivation of a set of algebraic expressions (non-visual evidence of proof search).
- Instrumental (+++) scores: The students have the abilities related to the (+) scores, but they show a deep understanding of the necessity of proof. This is reflected in the fact that the students follow different proof search strategies. The achievements in the proof search have valuable mathematical content. For instance, these students apply complex figural inferences (that even may have the status of a proof without words (Nelsen, 1993)). They search for algebraic-geometric invariants combining the simultaneous use of the geometric window, the algebraic window and the dragging tool.

We define in the following table the indicators for the different scores concerning the instrumental competence (table 3.22).

<table>
<thead>
<tr>
<th>Instrumental competence</th>
<th>(-) scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1. Lack of necessity of proving</td>
</tr>
<tr>
<td></td>
<td>2. Systematic lack of conjectures</td>
</tr>
<tr>
<td></td>
<td>3. Lack of validation of conjectures (conjectures that comes directly from perception)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Instrumental competence</th>
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<tbody>
<tr>
<td></td>
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<td></td>
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<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Instrumental competence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
|                         | 2. Search for conjectures-lemma (equivalent to considering a
Deductive competence

We define the meaning of each score concerning the deductive competence, and we give some indicators for each score.

- Deductive (-) scores: *Naïve justifications*. The student is not aware of the different values of deductive aspects, such as accepted theorems and definitions.
- Deductive (o) scores: Partial awareness of the different values of the deductive aspects.
- Deductive (+) scores: The students understand the need for generic examples to prove universally quantified statements.
- Deductive (++) scores: The students are able to construct deductive proofs (*thought experiment*). For instance, they reduce the problem to key-lemmas, they have insight into the proof by contradiction.

We define in the following table the indicators for the scores concerning the structural competence (table 3.20).

### Table 3.22: Instrumental codes

| (+) scores | 1. Figural inference: visual proof  
2. Spontaneous conjecturing and self-initiated efforts to verify the conjectures deductively  
3. Search for elaborated conjectures-lemma (equivalent to considering a subproblem)  
4. Internalization of dragging as theoretical control. Dragging elements in a wide range through continuous transformation (change of orientation of a figure for instance)  
5. Search for invariants (algebraic-geometric). |
|鹳|鹳|鹳|鹳|鹳|

**Deductive competence**

1. Empirical approach to the establishment of the truth of a statement (*naïve justifications*)
2. Absolute lack of clarity concerning the functions of definitions, well-known properties, theorems and proofs

**(-) scores**

1. Explicit use of the logical form ‘if … then’ in the formulation and handling of conjectures, as well as implicit use of logical rules such as *modus ponens*.
2. Awareness of the distinction between implication and double equivalence, transitivity of the implication
3. Identification of tautologies
4. *Generic examples*

**(o) scores**

1. clarity concerning the functions of definitions, well-known properties, theorems and proofs
2. Awareness of the different formal strategies of proving.
contradiction, by equivalence for instance.
3. Elaborated proofs (thought experiment)
4. Reduction of the problem to a key-lemma, or to a series of key-lemmas

| Table 3.20: Deductive codes |

We exemplify, which some paragraphs of protocol, the coding system considered for each geometric competence.

**Visualization competence**

We assign the code (-) to a student who has been misled by the orientation of the figure to construct the exterior height of a triangle in the third problem (problem of the median to be solved in a technological environment).

We assign the code (o) to a student who in the problem of the median, considers an isosceles triangle and understands that the property ‘the median splits the triangle into two congruent triangles does not hold for other triangles’. He understands the distinction between drawing and figure in this particular case. Nevertheless, he does not visualize that in the case of a general triangle the two inner triangles share the height, and moreover he does not consider other triangles. We exemplify this fact in the following paragraph:

**Student:** Podries resoldre aquest problema d’una altra manera?
**Tutor:** No. No, sé com ho podria fer.
**Student:** Podries utilitzar una estrategia basada en descomposició d’àrees?
**Tutor:** És que aquesta línia [median] parteix el triangle per la meitat [congruent triangles], però això depèn del triangle [the triangle constructed is isosceles].

We assign the code (+) to a student who in the fourth problem visualizes, that the addition of the areas of both triangles is constant. He states ‘L’únic canvi és que un perd base i l’altre en guanya, però les altures, en principi, es conserven.’ He visualizes the variation of the point P along the base, and he infers (figural inference) that the addition of both areas remain constant. This problem has to be solved with paper-and-pencil.

We assign the code (++) to a student who in the fourth problem visualizes, through reconfigurative operative apprehension that the addition of the areas of both triangles is constant. In this case, we assign the code (++) because the student modifies the initial configuration by transforming both triangles into an equivalent triangle, as we can see in the following paragraph:

**Student:** Podem agafar aquest triangle i dir que és un sol triangle imaginari de base AC.
**Tutor:** Què vols dir?
**Student:** Bé, si tinge dos espais [both triangles] i dic que només en considero un.

**Structural competence**

In the resolution of the second problem (problem of the scaled triangles), a student has difficulties in understanding the logical structure of the problem’s statement. The following paragraph shows that the student did not understand the statement of the problem at first, but he then corrected partially the formulation:
We assign the code (o) to this paragraph because the student understands partially the logical structure of the problem. Nevertheless, he still considers that the property EM=MF is only valid for a particular point P which verifies the property (trisection of the base of the triangle).

Another student considers that he can obtain the concrete values of the areas of both rectangles in the root problem by solving a linear system of two equations and two unknowns. He does not understand the logical structure of the problem (the point P is any point of the diagonal). We code the corresponding paragraph with a (-) score.

**Instrumental competence**

In the third problem, which has a verbal statement and has to be solved in a technological environment, one of the students states:

Student: Com que no diu res puc agafar un triangle rectangle.  
Tutor: Si, però el raonament que facis ha s'ha de poder generalitzar a altres triangles.  
Student: [The student constructs then an isosceles triangle and he does not drag the vertices of the initial triangle during the resolution process]

This paragraph provides evidence of a lack of understanding of the logical structure of the problem, and also gives us insight into a lack of understanding about the instrumental competence. He considers that he can justify the property only for the case of rectangle triangles (pragmatic justification). We code this paragraph with a (-) score.

In the second problem, the scaled triangles problem, which has to be solved in a technological environment, one of the students proposes a naïve justification. We assign to this kind of pragmatic justification the (o) score, which reveals partial insight into the competence. The student states: ‘Com ho demostres si totes les lletres són diferents? Puc agafar nombres d’aquí [measure tools of GeoGebra] per comprovar-ho? We observe that the student is aware of the fact that the justification proposed (naïve justification) is not valid. We observe that he mentions the verb ‘comprovar’ instead of ‘demostrar’. Nevertheless, we do not have enough evidence that he understands the meaning of proof.

**Deductive competence**

In the fourth problem, problem of the quadrilateral, that has to be solved with paper-and-pencil, one of the students constructs the following conceptual justification (Balacheff, 1998).

Student: Suposem que les àrees són iguals  
Student: Per Thales, jo sé que MN és un mig de AC

Aleix: Ara igualo les àrees i vaig simplificant, a veure que queda... \[ \frac{AC}{2}h = 2\left(\frac{MN}{2}\right) \]
Student: [He obtains \( MN = \frac{1}{2} AC \)]

Student: Com això per Thales és correcte, i simplificant les fòrmules de les àrees surt el mateix, llavors és correcte.

We code this paragraph with the (++) score, because it gives insight into a high level of deductive competence.

For each problem and for each student, we condense the information of the coding system in a table (see table 3.21). We condense in this table the examples presented in this section to illustrate the way in which we condense the coding information, despite the fact that these examples correspond to different students and to different problems.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Lines corresponding to the paragraph that gives insight into the competence for each score and for each competence</th>
<th>Protocol lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(-) The student is misled by the orientation of the figure (o) Distinction between drawing and figure but the student does not consider changing the figure to generalize the properties observed (+) Dynamic visualization (linear element: bases of triangles) (+++) reconfigurative operative apprehension (transformation of two triangles in one equivalent triangle)</td>
<td></td>
</tr>
<tr>
<td>Structural</td>
<td>(-) The student does not understand the logical structure of the problem (P any point of the median) (o) The student understands partially the logical structure of the problem (one of the quantifiers)</td>
<td></td>
</tr>
<tr>
<td>Instrumental</td>
<td>(-) The student considers only particular cases (o) The student is aware of the fact that the justification proposed (naïve) is not valid</td>
<td></td>
</tr>
<tr>
<td>Deduction</td>
<td>(++) conceptual justification</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.21: Competences’ coding system for a problem

We also report the acquisition degree of instrumentation: we analyse the development of instrumented schemes for the GeoGebra problems.

We interpret the difficulties encountered by the student, the acquisition degrees of each competence, the instrumentation taking into account the expected effect of the problem, the starting level of the student and the role of both the tutor and the environment.

In the last phase, we consider the transitions of competences between two consecutive micro-cycles. We compare and interpret the differences in the acquisition degrees of competence between both problems.

### 3.2.4 Transitions between the micro-cycles

To identify the transitions between the micro-cycles (problems), we show again the scores that refer to the acquisition degrees of the geometrical competences for each problem with the purpose of comparing the different acquisition degrees for each competence and for each problem (Tables 3.22 and 3.23) It is important to remark that we consider this comparison from a qualitative point of view, considering differences in the scores assigned for each competence. This comparison makes sense, since we focus on problems that are similar. We transfer this problem’s similarity to the fact that they
share resolution strategies (basic space) and the number of steps for solving these problems is similar. We describe the transitions for each competence between each micro-cycle giving explanations for the differences between the acquisition degrees.

**Feed-forward from the second teaching experiment**

<table>
<thead>
<tr>
<th>Competences</th>
<th>lines corresponding to the paragraph that gives insight into the competence for each score and for each competence</th>
<th>Protocol lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Structural</td>
<td></td>
<td></td>
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<tr>
<td>Instrumental</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deduction</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.22: Competences’ coding system for the second problem

**Feed-forward from the third teaching experiment**

<table>
<thead>
<tr>
<th>Competences</th>
<th>lines corresponding to the paragraph that gives insight into the competence for each score and for each competence</th>
<th>Protocol lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Structural</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Instrumental</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deduction</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.23: Competences’ coding system for the third problem

**3.2.5 Characterization of the learning trajectories**

Finally, we characterize the learning trajectories in terms of the transitions between the micro-cycles concerning the different competences, the tutor’s orchestration and the synergy of environments. We reflect on the findings, consider possible modifications of the proposed problems and identify the activities that foster the expected insight into geometrical competences. This is the feed-forward of the local cycle.

In the following chapter, we present the analysis of data considering the phases explained in this section. Nevertheless, we present the phases in a different order. We start with the third phase (3.6.3 Summary of the protocol: applying theory to practice) and we present then the fourth and the fifth phases. The first two phases are not included because these phases are carried out in order to facilitate the analysis, but we present again the relevant information in the following phases.
4. Learning trajectories: profiles, feed-forward and transitions

In this section we analyse three students’ learning trajectories according to the qualitative analysis detailed in the third chapter (3. Design research and Methodology). For each student, we analyse each micro-cycle and the transitions between each micro-cycle. These students, who are Marta, Guillem and Aleix, have followed the third itinerary of problems. In this section we analyse the local cycle, which consists of four consecutive problems of the third itinerary, and which are: the root problem, the scaled triangles problem, the median problem and the quadrilateral problem.

4.1 Analysis of the resolution process of the root problem

Firstly, we summarize and interpret the students’ resolution processes of the root problem, and secondly we report the feedback concerning the root problem for each student.

4.1.1 Summary of the protocol: applying theory to practice

In the following sections, we analyse the students’ resolution processes of the root problem with paper-and-pencil. We summarize and interpret the resolution processes of Marta, Guillem and Aleix.

4.1.1.1 Summary of Marta’s protocol

For the resolution of the root problem, Marta tries first an algebraic approach based on obtaining equality of ratios and applying the area formula of a rectangle. After a local evaluation phase, she changes the first resolution approach and considers an equivalent problem. She tries then to state a conjecture about the relation of areas, but she is persuaded that these areas can not be equal. The tutor helps Marta to state a conjecture about the areas’ relation. This message is relevant, since Marta reacts by trying to deduce that both areas are equal. This fosters a resolution strategy based on applying equicomplementary dissection rules, as we will see in the following paragraphs.

After the familiarization phase, in which she identifies the objective of the problem and writes on the figure the given lengths of the sides AD and AB (Figures 4.1 and 4.2), Marta starts an analysis/exploration phase. Marta’s statement (line 3) marks the beginning of this phase, as we can see in the paragraph below.

![Figure 4.1: Figure associated to the problem]
Marta interprets that she needs Thales theorem to solve the problem. She may focus attention on the fact that the lines through E are parallel to the sides of the rectangle. We observe in the video records that she points at the segments AN and AB and then AM and DP (Figure 4.1). She tries to apply Thales theorem in the triangles (usual configuration) ANE and ABC and analogously in the triangles AME and ADC (operative apprehension). By applying Thales theorem in two steps, she may have obtained the following equality of ratios \[ \frac{AN}{AB} = \frac{AM}{MD} \] and then reduced it to the equality of areas of the rectangles ABOM and ANPD, which is equivalent to the equality of areas of the rectangles considered in the statement.

Nevertheless, Marta abandons this strategy. She states, ‘a veure, és el rectangle NEBO i MEPD..., uf!’ We conjecture that Marta associates Thales theorem only with the comparison of similar triangles in a usual configuration. We base this conjecture on the fact that she may obtain the equality of ratios AN: NB=AM: MD (Figure 4.1) by applying Thales theorem in two steps. Moreover, she does not discern the triangles AEM and EOC, which are also in the Thales configuration and allow us to obtain an equality of ratios, which is equivalent to the equality of areas. After a silence (line 6), Marta tries again to relate the data and the objective of the problem. She states: ‘Només necessito obtenir AM i ME ...perqué això ja ho sé [angle <E]. O sigui, l’angle ha de ser el mateix’ [angle in C and angle in E are equal in the triangles AEM and ACD]. Despite not mentioning it, she uses the fact that the inner angles formed by a secant line to two parallel lines are equal. We find it relevant that she does not try to apply Pythagoras theorem to obtain the diagonal AC. This may be due to the fact that she focuses on applying Thales theorem to obtain information about the sides AM and ME. She may consider that the length of the diagonal AC is superfluous, as she can not derive the length of AE. Moreover, Marta mentions in the resolution of the second problem that Thales theorem and Pythagoras theorem are the main tools to tackle triangles problems. She states ‘Triangles és Thales o Pitágoras’.

Marta observes that she does not know the length of the segment AE (line 8) and decides to represent the lengths of AM and AN with the letters x and y (Figure 4.3). She tries to obtain the expression of both areas, relating the sides and the angles of the triangles) in order to find the relation between the areas, she abandons Thales theorem to apply trigonometry of the right-angled triangle, as we can see in the following paragraph.
11. Marta: Clar no cal ficar els números. O sigui, puc posar que aquest rectangle és... el...
12. Marta: O sigui allàent el cosinus que seria costat contigu [ME] per hipotenusa tinc aquest costat contigu i multiplico per 6 menys... [6-y, lenght of AM]
13. Marta: La tangent de l’angle [E] seria 6 partit per 8...
14. Marta: però jo que sé... pot ser 3[x] partit per 4 [y] en l’altre triangle [AME] ...
15. Marta: La tangent és sis entre vuit que és tres entre quatre que dona 0.75. És pot obtenir l’angle amb la funció inversa que és arctan... o sigui l’arctan de 6 partit per 8 donaria l’angle
16. Marta: És que, és molt difícil! És que potser no vaig bé...

**Figure 4.2:** Expression of the sides AM and AN in terms of unknowns

This paragraph shows also that Marta understands partially the logical structure of the problem. She considers the lengths of AM and AN as unknowns, but she does not interpret these unknowns as parameters. She mentions that the values of x and y may be three and four units as result of considering an equivalent fraction to $\frac{6}{8}$, but she does not visualize that this corresponds to the particular case in which E is the midpoint of the diagonal. If she had considered this geometric particularization, she may have conjectured the relation of areas.

With the last intervention (line 16), Marta starts a local evaluation phase. She asks for a validation message from the tutor, which gives her the following cognitive message of level one: ‘Hi ha diferents maneres de resoldre el problema. Has trobat resultats que poden ser útils’ (line 18 of the entire protocol). Marta reacts to this message by considering again Thales theorem. She abandons the trigonometric strategy and selects again the use of Thales theorem. She states ‘És que això és Thales!’ (line 19 of the entire protocol).

She starts again an analysis/exploration phase, but she has difficulties in stating a conjecture about the relation of the areas, as we can see in the following paragraph. She mentions that she should find the area of the rectangle AMNE (line 21). This fact gives us insight into an incomplete understanding of the logical structure of the problem. We wonder if she understands the dynamic structure of problem. We observe also that she is focusing on an equivalent problem (comparison between the area of the rectangles ABOM and ANPD), but the fact that she does not state a conjecture makes this resolution process difficult.

21. Marta: O sigui... per trobar... He de trobar això i això per tenir la relació...
22. Marta: he de trobar l’ àrea d’aquest [AMNE] i restar-la dels dos costats.[Figure 4.3]
23. Marta: Jo diria que és, o sigui la relació és l’àrea de tot això menys això i l’àrea de tot això menys això... [equivalent problem based on comparing the areas of both rectangles ABOM and ANPD but she does not state a conjecture].
24. Tutor: et refereixes a que tenen la mateixa àrea? [Cognitive message of level three]
25. Marta: No! no tenen la mateixa àrea…
26. Marta: Ah! Potser sí!
27. Marta: Tenen la mateixa àrea?
28. Tutor: per què creus que podrien tenir la mateixa àrea?

**Figure 4.3**: Equivalent problem: ‘l’àrea de tot això menys això i l’àrea de tot això menys això’

The tutor’s message (line 24) is a cognitive message of level three that has a relevant influence on Marta’s resolution process. This message, which is part of the didactical performance, reveals to Marta the relation between the areas. It would have been better for the tutor to have given a heuristic message of level one to help Marta to state a conjecture (to be coherent with the criteria for assigning messages). In this case, the tutor interprets wrongly Marta’s statement (line 23), and he interprets that she has conjectured the equality of both areas. This message leads Marta to consider as strategy based on equicomplementary dissection rules.

Marta starts a planning/execution phase and tries to discern congruent triangles. We observe the coordination between operative and discursive apprehensions. This coordination process provides the ‘idea’ of how to solve the problem deductively. This kind of process is defined as truncation by Torregrosa and Quesada (2008).

Marta discerns three inner rectangles that share the diagonal (operative apprehension), and she obtains the key idea of splitting each rectangle in two congruent triangles. She associates the property of the diagonal that splits a rectangle in equivalent triangles. She then applies the properties of the area function as a measure function (additive properties), as we can see in the following paragraph.

27. Marta: Mira! Tenen la mateixa àrea! Aquests dos triangles són iguals [AME and ANE] i aquests dos són iguals també [EPC and EOC].
28. Tutor: Com ho saps?
29. Marta: Clar! És la meitat del rectangle
30. Tutor: Si, ja veig
31. Marta: Si restes dues àrees iguals de dues àrees iguals et queda la mateixa àrea.

Finally, Marta has solved the first problem with a strategy based on decomposition of the rectangle into congruent triangles. The tutor asks Marta what the key idea has been for deducing the equality of areas. She states ‘He pensat, la diagonal ho parteix per la meitat, per tant un cantó ha de ser igual que l’altre.’

**4.1.1.2 Summary of Guillem’s protocol**
For the resolution of the root problem, Guillem tries first an algebraic approach that shows a lack of understanding concerning the logical structure of the problem. He considers the lengths ME and NE as unknowns, but he does not understand that these lengths are also parameters that depend on the length of AE. The tutor has a key role; he
fosters a geometric resolution based on splitting the rectangle into congruent triangles and the partial understanding of the logical structure of the problem.

After the familiarization phase, Guillem starts an exploration phase trying to derive new data from the data of the problem. He applies firstly Pythagoras theorem and then tries to obtain the relations between the sides and the angles of the right-angled triangles AME and ADC (Figure 4.4). He extracts these triangles from the initial configuration (operative apprehension) but he does not mention that these triangles are similar. This reconfigurative process obstructs the consideration of the resolution approach based on equicomplementary dissection rules, because he focuses only on one side of the initial figure (Figures 4.4 and 4.5).

Guillem has in mind the strategy of finding the concrete lengths of the sides AM $[x]$ and ME $[y]$, as we can see in the following paragraph:

6. Guillem: Es pot obtenir la tangent en un triangle. La relació és la mateixa en aquest triangle [AME] que en aquest [ADC]
7. Guillem: la relació és la mateixa [similar triangles], però…
8. Guillem:… podria utilitzar proporcionalitat, una regla de tres…però no conec això… [AE]…

[Guillem stares at the figure in silence for a while]

As he notices that there are two unknowns to derive the area of both rectangles, he tries then to obtain a system of two linear equations. This paragraph shows that he does not understand the logical structure of the problem (quantifier for the point E and dependence of the points M and N). To achieve his goal, he extracts three similar right-angled triangles, which are AME, EPC and ADC (Figure 4.4). He does not mention that the triangles are similar. Implicitly, he uses that a secant line to two parallel lines forms equal inner angles and trigonometry of the right-angled triangle.

He starts an execution phase by trying to solve the system of equations formed from the equality of ratios, as we can see in the figure below (Figure 4.6). He notices then that he gets the same equation and he states: ‘Et queda lo mateix!’ He is not able to interpret that the system has infinite solutions (all the points of the diagonal). Moreover, he has enough data to find the relation between the areas:

$$\frac{x}{y} = \frac{6-x}{8-y} \Rightarrow x(8-y) = y(6-x)$$

The fact that Guillem does not try to state any conjecture of the relation between the areas...
also makes it more difficult for Guillem to discern the equality between the areas in the previous equality of ratios. He does not interpret the expression obtained \( x(8 - y) = y(6 - x) \) as the equality between the areas of both rectangles. He does not consider that this is a solution because he expects a numerical expression for these areas.

![Figure 4.6: Right-angled triangles extracted and equality of ratios (We have inserted the labels M, E and P to name the rectangles considered by Guillem)](image)

This fragment shows that Guillem has a lack of conceptual knowledge concerning systems of equations, and also that Guillem does not understand the logical structure of the problem. We code this fragment with a (-) score concerning the structural competence. As Guillem is lost, the tutor gives him a cognitive message of level two that has a relevant effect. Guillem reacts by trying to state a conjecture about the relation of the areas, as we can see in the following paragraph.

13. Guillem: Et queda lo mateix! [He gets only one equation (Figure 4.6)]
14. [He stares at the figure in silence. He is lost]
15. Tutor: No calen dades concretes per resoldre aquest problema [cognitive message of level two]
16. Guillem: No sé...Em sembla que tenen la mateixa àrea
17. Tutor: Perquè creus que tenen la mateixa area?
18. Guillem: No sé,...perquè lo que perd un de un de llargada o guanya d’amplada

The purpose of the tutor’s message (a priori designed message) is to focus students’ attention on the dynamic structure of the problem. The equality of areas holds for any point E of the diagonal. Guillem reacts to the message by trying to state a conjecture, but he does not interpret the solution obtained for the system of equations.

We code the above paragraph with a (o) score, as it reveals partial insight into the instrumental competence. For the first time Guillem tries to state a conjecture. Nevertheless, he validates the conjecture with a false property. He states ‘lo que perd un de llargada o guanya d’amplada’. He observes through perceptual approach in the grid that the base of the first rectangle is larger than the base of the second rectangle, but this difference is compensated by the rectangles’ heights. Implicitly, he considers an additive property that is not fullfilled for the areas. Implicitly, he applies the following wrong
property: \((a - \Delta) \cdot b = a \cdot (b - \Delta)\). We conjecture that he makes the analogy of the additive property fulfilled for the perimeter: \(2(a + (b - \Delta)) = 2((a - \Delta) + b)\). Nevertheless, this is a good attempt that has value as insight into the structural competence. Through a *figural inference* he conjectures that the rectangles’ areas are equal (he relates the bases and the heights). Guillem starts an exploration phase by trying to prove the equality of areas. As there is no feedback from the student, the tutor gives him a message of level two:

21. Tutor: recorda les propietats de la diagonal d’un rectangle
22. Guillem: Si, ja ho veig…aquests triangles tenen la mateixa àrea, el mateix que per un fa d’alçada a l’altre fa de base i a l’inversa.. [He express the area of each rectangle using equicomplementary dissection rules and he equals both expressions]
23. Guillem: puc simplificar..[Guillem tries to solve a equation that results from supposing that the areas are equal] (Figure 4.7)
24. Guillem: Queda zero igual a zero![difficulty to understand equivalences, he may expect again a concret value for the areas]
25. [Guillem stares at the worksheet in silence]
26. Guillem: clar, és que són iguals....
27. Guillem: Al dividir-ho dona 1 que vol dir que són iguals [difficulty to visualize equicomplementary dissection rules]

Guillem deduces that the remaining areas are equal, since he obtains an equivalence. Guillem uses the area formula of the triangle to prove that the areas are equal, as we can see in the figure below (Figure 4.7).

\[
\frac{6 \cdot 8}{2} = 24
\]
\[
\frac{24 - \frac{(6 - x)(8 - y)}{2}}{2} \quad \frac{24 - \frac{(6 - x)(8 - y)}{2}}{2} = \frac{24 - \frac{(6 - x)(8 - y)}{2}}{2}
\]

**Figure 4.7:** Strategy based on decomposition of the rectangle in equal triangles

### 4.1.1.3 Summary of Aleix’s protocol
In the resolution of the first problem, Aleix focuses on a resolution approach based on discerning similar rectangles and applying properties of area function as a measure function. Finally, he discerns congruent triangles and applies the strategy based on equicomplementary dissection rules.
In the familiarization phase, Aleix reads the statement of the problem and marks segments of equal length on the rectangles whose areas have to be compared (Figure 4.8). A relevant aspect is that he does not pay attention to the concrete lengths of the sides of the rectangle ABCD (AB = 8 units, AC = 6 units) and starts the exploration phase by conjecturing the relation between both areas. He states ‘Llavors, he de justificar que els dos rectangles, aquest i aquest són iguals? He refers to the fact that both rectangles have the same area. Despite the fact that Aleix has in mind the property of the diagonal of a rectangle (it splits the rectangle into congruent triangles), and he discerns the inner rectangles ANME and EOPC, he does not apply equicomplementary dissection rules. He tries to discern similar rectangles, as we can see in the paragraph below.

We conjecture that he may try to obtain a relation between the areas of the inner rectangles and the outside rectangle, or he may try to prove that the rectangles to compare are similar and then obtain that the similarity factor is one. We base our conjecture on the fact that Aleix states, by analogy with the congruence criteria AAA for triangles, that the inner rectangles AMNE and EPCO are similar to the outside rectangle ABCD using the analog criteria AAAA, as we can see in the paragraph below.

3. Tutor: Perquè creus que tenen la mateixa àrea?
4. Aleix: Pues,...ABC és el rectangle gran i AC la recta que és la diagonal per definició i la diagonal divideix al rectangle en dos triangles iguals...llavors...
5. Aleix: ...aquí també hi ha un rectangle...[EPCO]
6. Aleix: EPCO [Figure 4.8] és un rectangle similar al gran perquè té tots els costats paral·lels.

7. 1 Tutor: Estàs segur? [Cognitive message of level one]
8. 1 Tutor: És sufficient que els costats siguin paral·lels per tal que les figures siguin similars?
9. Aleix: Sí. El costat d’un multiplicat pel costat de l’altre ens donarà el mateix coeficient. Si els angles són iguals, la forma ha de ser la mateixa
11. Aleix: però, vale... [He remains in silence looking at the figure]
12. Tutor: A què et refereixes?
In this case, the tutor tries to encourage the student to check the validity of the analogy and asks the student: *Estàs segur?* As the student does not react, and answers ‘*Si, si tots els angles són iguals les figures tenen la mateixa forma*’ the tutor gives him a cognitive message of level two (lines 7.1 and 7.2). Finally, Aleix reacts to the following cognitive message and obtains the congruence criteria for the rectangles. Implicitly, he considers the similarity ratio, but he has difficulties in expressing it properly. He states “*El costat d’un multiplicat pel costat de l’altre ens donarà el mateix coeficient*” (line 8). He may consider the ratio NE: EP=ME:EO, which is equivalent to the equality of areas, but he does not mention this fact. Finally, Aleix visualizes the congruent triangles and applies equicomplementary dissection rules, as we can see in the following paragraph.

14. Aleix: EPCO és proporcional al gran i també hi ha un rectangle petit ANEM.
15. Aleix: el rectangle petit també és proporcional al gran i per suposat al mitjà [EPCO]
16. Aleix: llavors...com la diagonal en tots els casos és la mateixa, llavors, a cada part, en cada triangle BAC i ACD aquests petits triangles són iguals [AME i ANE; EPC i EOC]
17. Aleix: l’espai que ens queda també ha de ser igual [Figure 4.10]

We code this paragraph with (+) store concerning the visualization, structural and deductive competente.

![Image of Aleix's worksheet](image)

**Figure 4.10:** Aleix’s worksheet (root problem)

### 4.1.2 Feed-forward concerning the local HLT: root problem

In this section we report the feedback concerning the root problem for the three students.

#### 4.1.2.1 Feed-forward concerning the root problem: the case of Marta
Marta understands partially the logical structure of the root problem. At first she tries to
determine the lengths of the sides AM and ME, but she realizes that she can not derive
these lengths if she does not have the length of the segment AE. The following
paragraph reveals partial insight into the structural competence and we code it with a (o)
score. She understands partially the logical structure of the problem.

7. Marta: Només necessito obtenir AM i ME …perquè això ja ho sé [angle <E]
O sigui, l’angle ha de ser el mateix [angle in C and angle in E are equal in the triangles AEM and
ACD]
8. Marta: però…clar no sé això… [AE]
9. Marta: Bé, he de buscar la relació no cal que sàpiga…
11. Marta: Clar no cal ficar els números. O sigui, puc posar que aquest rectangle és…el..

Nevertheless, she does not understand the dynamical structure of the problem (E can be
any point of the diagonal), and this fact obstructs the consideration of a variation of the
point E along the diagonal. The figure of the statement inserted in a grid obstructs the
understanding of the dynamical structure of the problem.

Despite the fact that she mentions a particular case, as we observe in the following
paragraph, she does not use this case to state a conjecture about the relation between the
rectangles’ areas and later she abandons the solving approach.

12. Marta: O sigui aïllant el cosinus que seria costat contigu [ME] per hipotenusa tinc aquest
costat contigu i multiplico per 6 menys… [6-y, lenght of AM]
13. Marta: La tangent de l’angle [E] seria 6 partit per 8…
14. Marta: però jo que sé…pot ser 3[x] partit per 4 [y] en l’altre triangle [AME]…
15. Marta: La tangent és sis entre vuit que és tres entre quatre que dona 0.75. Es pot obtenir
l’angle amb la funció inversa que és arctan…o sigui l’arctan de 6 partit per 8 donaria l’angle

She does not visualize the dynamic variation of the point E along the diagonal. If she had
considered the particular case x=3 and y=4 (E midpoint of the diagonal), she may have
conjectured a relation between the areas. Marta does not try to conjecture a relation and
this leads her to consider algebraic strategies based on considering theorems to relate the
sides and the angles of the right-angled triangles and on applying the area formula of
each rectangle. We code this paragraph with a (-) score concerning the visualization
competence, as she does not visualize this particular case (algebraic particularization).

Marta is reticent to state conjectures about the areas’ relation. She has difficulties in
visualizing the equality of areas when the tutor helps Marta to state the conjecture, as we
can see in the following paragraph. We code this paragraph with a (-) score for the
visualization competence.

23. Tutor: et refereixes a que tenen la mateixa àrea? [Cognitive message of level three]
24..Marta: No! no tenen la mateixa àrea…
25. Marta: Ah! Potser sí!
26. Marta: Tenen la mateixa àrea?

The main obstacle in the resolution process is the difficulty in stating a conjecture about
the relation of both areas, and the fact that she does not consider stating a conjecture. This
is related to the instrumental competence that we code with a (-) score.
The tutor’s message (line 23) marks a transition in the resolution process. Marta tries then to discern congruent triangles and she does not focus any more on resolution approaches based on obtaining equality of ratios. This leads Marta to visualize the inner congruent rectangles and to justify deductively that both areas are equal, as we can see in the following paragraph:

29. Marta: Mira! Tenen la mateixa àrea! Aquests dos triangles són iguals \([\text{AME and ANE}]\) i aquests dos són iguals també \([\text{EPC and EOC}]\).
30. Tutor: Com ho saps?
31. Marta: Clar! És la meitat del rectangle
32. Tutor: Si, ja veig
33. Marta: Si restes dues àrees iguals de dues àrees iguals et queda la mateixa àrea.

We code this paragraph with a (+) score concerning the visualization competence, the structural competence and the deductive competence.

We condense in the following table (Table 4.1) the scores that refer to the acquisition of geometric competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Root problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(-) She does not visualize neither the variation of the point E along the diagonal, nor the midpoint particular case (lines 12-15)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulties in visualizing the equality of areas (lines 22-18)</td>
</tr>
<tr>
<td></td>
<td>(+) She discerns inner congruent triangles (lines 29-33)</td>
</tr>
<tr>
<td>Structural</td>
<td>(o) She understands partially the logical structure of the problem (lines 7-11)</td>
</tr>
<tr>
<td></td>
<td>(+) She considers the properties of the diagonal and the area function as a measure function (lines 29-33)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(-) She does not consider stating conjectures about the relation of areas</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) She produces a pragmatic justification (generic example) but she is not able to write it down in the worksheet (lines 29-33)</td>
</tr>
</tbody>
</table>

Table 4.1: Competences coding system (Marta)

This problem helps Marta to develop awareness of the basic properties of the area in the particular case of congruent figures – if you add or subtract equal areas you get equal areas. At first, she focuses only on applying Thales theorem, and this fact obstructs the visualization of other configurations (congruent triangles). Moreover, she has difficulties in applying Thales theorem in two-steps, as she is not clear about the relation between the areas. Also she has difficulties in visualizing triangles in a non-usual Thales configuration.

Taking into account that the Marta has solved the first problem using a resolution approach based on comparing areas (equivalent due to complementary dissection rules), the tutor proposes problems from the third itinerary to Marta.

4.1.2.2 Feed-forward concerning the root problem: the case of Guillem

We condense in the following table the scores concerning the different competences (visual, structural, instrumental, deductive). In the resolution of this problem, the most relevant difficulty is the understanding of the logical structure of the problem. Guillem tries to obtain the concrete lengths of the sides of the interior rectangle ANME. This fact obstructs the resolution process. Moreover, Guillem does not try to conjecture the areas’ relation. Guillem reacts to the tutor message by abandoning the algebraic approach, based on obtaining equality of ratios, and by focusing on the approach based on
equicomplementary dissection rules. Nevertheless, Guillem does not visualize this relation and tries to justify this fact by using the formula of the areas of a triangle. We conjecture that he is not aware of the fact that he can not obtain the concrete values of the rectangles areas. Finally, he proves the equality of areas.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the root problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(+) Operative apprehension: he extracts three similar right-angled triangles (Figure 4.25)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing equivalent figures and equicomplementary dissection rules (lines 21-27)</td>
</tr>
<tr>
<td>Structural</td>
<td>(-) He does not understand the logical structure of the problem (lines 6-8, 13-18)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(0) Figural inference based on analogy (lines 13-18)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(0) Deductive steps to prove the equality of areas (lines 21-22 and Figure 4.26)</td>
</tr>
</tbody>
</table>

Table 4.2: Competences coding system (Guillem)

4.1.2.3 Feed-forward concerning the root problem: the case of Aleix
In the resolution of the root problem, Aleix develops awareness of the fact that the congruence criteria AAA for triangles can not be extended as AAAA criteria for other polygons. Moreover, Aleix understands the logical structure of the problem and is aware of the properties of the area function as a measure function. In the following table we present the scores assigned to each competence.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Exploration with the root problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(+) Operative apprehension (he extracts congruent triangles)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (he extracts similar rectangles)</td>
</tr>
<tr>
<td>Structural</td>
<td>(+++) Understands the logical structure of the problem</td>
</tr>
<tr>
<td></td>
<td>(+) Analogy criteria AAA for rectangles. He understands the criteria</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension: property of the diagonal (it splits a triangle into two equivalent triangles), properties of the area function as measure function (resolution process)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(+) Conjecturing (line 2)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) Deductive reasoning to prove the conjecture.</td>
</tr>
</tbody>
</table>

Table 4.3: Competences coding system for the root problem (Aleix)

As we see it, we should consider modifications of this problem to foster students’ acquisition of competences. For instance, we may consider a generalization of this problem (parallelograms). We may also ask for the relation between the areas of both inner rectangles and the area of the rectangle ABCD.

4.2 Analysis of the resolution process of the problems
Firstly, we summarize and interpret the student’s resolution processes of the scaled triangles problem, the median problem and the quadrilateral problem, and secondly we report the feedback concerning these problems for each student.

4.2.1 Summary of the protocol: applying theory to practice
In this section we analyse the three students’ resolution processes of the scaled triangles problem, the median problem and the quadrilateral problem. We consider firstly Marta’s
resolution processes, then Guillem’s resolution processes and finally Aleix’s resolution process.

4.2.1.1 Summary of the protocol: The case of Marta

a) Marta’s resolution process of the scaled triangles problem

In the familiarization phase, Marta tries first to reproduce the figure of the problem’s statement with GeoGebra, but she has difficulties in understanding P as “any point of the median”. As in the root problem, she considers P as a concrete point of the segment. To achieve her goal, Marta tries first to define the point E as the midpoint of the segment BM. She states ‘Però és que aquí sí que està al mig... es veu en el dibuix que E i F són punts mitjos.’

Figure 4.11: Figure included in the statement of the problem

She does not distinguish spatial properties from geometrical properties. This misconception may be due to the fact that the figure of the statement is represented on a grid, and Marta interprets that the triangle ABC is defined with the lengths of its segments and the point P is a specific point on the median. The tutor suggests that Marta read again the statement of the problem, and gives her a conceptual message of level 2 (line 15), suggesting to Marta the construction of the point P. Nevertheless, Marta has not understood that P is any point in the median, for the first question. This misconception leads Marta to consider the second question of the problem (lines 10-12).

10. Marta: Ah, no! Mira. On hem de situar P per obtenir BE = EF = FC. Llavors no pot estar al punt mig....
11. Marta: Aquestes paral·leles han d’estar més avall. He de fer la mateixa figura pels dos apartats [She considers the same position of the point P for both questions: unique point P that verifies EM=MF and trisection of the segment BC]
12. Marta: Divideixo en tres parts... [the base BC of the triangle] i ja sé on està P.
13. Tutor: Com ho dividiries en tres parts iguales?[Figure 4.11]
14. Marta: Si això és 6...2, 2 i 2....o faria el punt mig 2 vegades [algebraic-geometric visualization difficulty]
15. Tutor: Llegeix l’enunciat del problema. Els punts E i F de què depenen?
16. Marta: de la paral·lela.... Depenen del punt P.
17. [Marta tries to construct the parallel lines before constructing the point P. This technical obstacle may be due to a lack of understanding of the logical structure of the problem]

She has technical difficulties with the instrumented technique “trisect a segment” (trisection of the segment [BC]). Marta drags the point P along the median to trisect
visually (she uses dragging for adjusting) the segment BC, using the grid on the screen, (Figure 4.11). She observes that she can not obtain the concrete length visually, thus she constructs the points E and F with the tool intersection and uses the tool distance with the help of the tutor. After, Marta drags the point P along the median to obtain BE =2u, observing the measure of BE on the geometric screen (Figure 4.11), but she has technical difficulties in interpreting the round-off error. The tutor suggests that Marta find first the relation between ED and DF (Figure 4.11), and she reacts by dragging the point P along the median to check that the property is conserved for other positions of the point P. By visualizing it with the dragging tool, Marta understands that the property ME=MF is conserved for all the points P on the median AM. As claimed by Strässer (1992), dragging offers mediation between drawing and figure and can only be used as such at the cost of an explicit introduction and analysis organized by the teacher. In this case, the tutor plays a role in guiding this mediation.

Finally, in the exploration phase, when trying to find elements to justify the conjecture, Marta observes visually on the geometric window that when dragging P along the median, the segments BE and ED “have the same relation” than the segments DP and PA (Figure 4.12). We observe here an instrumented scheme that consists in dragging combined with visual ratios comparison. This scheme requires the awareness of Thales theorem. In this case the exploration phase leads to the construction of a proof. Nevertheless, Marta loses track of the solving strategy due to difficulties with differences in the objects’ labels in both environments (paper-and-pencil and GeoGebra). She obtains expressions for EM and FM (M=D) and is not able to find the initial solving path. Marta proposes then a naïve justification, which consists on case-based reasoning: ‘Però com ho demostres si totes les lletres són diferents? Puc agafar nombres d’aquí per comprovar ho? [she proposes using measure tools]’ (line 38). The tutor reminds Marta about her initial strategy based on comparing the common side of both triangles (line 39).

![Figure 4.12: Marta’s construction (BE = EF = FC, M=D)](image)
The tutor will help Marta to deduce that the segments EM and FM have the same length (cognitive message of level two). The organization of Marta’s reasoning process is the following:

Dynamic Figure

+ Tutor’s message

\[
\frac{BM}{EM} = \frac{AM}{PM}
\]

\[
\frac{BM}{EM} = \frac{MC}{MF} \quad \text{[Midpoint of BC]}
\]

\[
EM = MC
\]

During the execution phase, Marta uses labels that appear in the figure of the statement of the problem (M instead of D). This may be due to technical difficulties with the option rename objects. She has obtained the expression

\[
\frac{BM}{EM} = \frac{MC}{MF}
\]

(Figure 4.13) by using Thales property twice, and is not convinced about the equality of the segments EM and MF, as she does not have in mind that M is the midpoint of the segment BC. She starts a verification phase. Marta states: ‘Però, podria ser que no fossin iguals i que fossin proporcionals...’. The tutor helps Marta to understand the statement of the problem by focusing Marta’s attention on the definition of the median (M is the midpoint of the segment BC, Figure 4.13).
For the second question of the problem, Marta considers again the figure on the screen. Marta drags the point P along the median AM to obtain \( BE = EF = FC = 2 \) units. This time she has overcome the previous technical difficulties. She observes the figure by trying to make a conjecture about the position of the point P in the median AM. She uses the tool distance to find relations between the segments AP and AM (Figure 4.12). We observe again the heuristic strategy of considering the problem solved. Although it is possible to apply Thales theorem, this time Marta does not consider this strategy immediately. She first tries to understand the statement of the problem and to state a conjecture. She may try to find in the figure, geometric properties of the point P that suggest a solving strategy.

Marta conjectures that the point P has to be the centroid of the triangle ABC, but she does not know how to justify this fact (Figure 4.14). As Hoyles and Healy (1999) state,
exploration of geometrical concepts using DGS helps the students to define and identify properties, but not necessarily leads to the construction of a proof. She remains silent for a while, observing the figure, but she does not consider the parallel lines. Finally, the tutor remarks that she should include again the parallel lines. She considers again the parallel lines (Figure 4.15) and starts an exploration phase.

![Figure 4.15: Parallel lines and centroid](image)

The last tutor’s message will suggest to Marta a solving path. Marta observes again the construction made with GeoGebra and she compares the lengths of the segments. With the help of the tutor, she finds out that $3 \cdot ED = BD$. Marta uses this property, with the help of the tutor, to justify that the points E and F trisect the base of the triangle. She does not validate the conjecture in other triangles (dragging option), and she is convinced by checking it in one figure. As stated by Soury-Lavergne (2008), it takes time before students apply the drag mode spontaneously. First, she begins to drag her diagrams locally and partially. In this case, it also may be due to the fact that the tutor gives Marta a message about the relation between these segments. Nevertheless, she discovers again this property with the use of measure tools applied to the segments.

Marta justifies empirically, naïve empiricism (Balacheff, 1998), that the medians trisect each other at the centroid. She uses first the fact that the three medians of any triangle all pass through one point as an accepted result. In this case, the use of GGB and the tutor’s orchestration suggest to Marta the strategy based on using the centroid properties.

The property of the centroid (point of trisection of the median) suggests to Marta a resolution strategy based on constructing parallel lines to BC through the centroid of the triangle to trisect the segments AB and AC (Figure 4.15). We can observe that she constructs parallel lines to (AC) through the centroid, but she does not notice that these lines should also trisect the side AB (Figure 4.8). If she had constructed these lines with GGB, she would have noticed that this geometric property leads to a resolution strategy which is not based on the properties of the centroid. It would have been better if the tutor had suggested that Marta constructed these lines with GGB.

We can see her resolution process in her worksheet (Figure 4.16). She does not mention Thales theorem, but she uses it implicitly to trisect the segment drawing parallel lines to the line AC through the centroid. She accepts the properties of the centroid (trisection of the median) based on a naïve justification (measure of segments on a concrete triangle). Also this property is more credible to Marta because the tutor validates the centroid
property. The tutor should have fostered the conceptual justification (Balacheff 1998) based on applying Thales theorem twice to trisect the segments. Marta is close to this resolution, as we can see in the drawing (Figure 4.16). Nevertheless, she does not visualize that the parallel lines to the side AC through E and F also trisect the side AB as we can see in the figure below. The oblique lines and the horizontal lines do not intersect each other on the side AB (Figure 4.16).

During the resolution process of this problem, Marta understands the dependency relationships of the problem’s elements. The use of GGB and tutor’s orchestration facilitate Marta’s understanding of the problem.

In the exploration phase, Marta identifies Thales property by using the dragging option combined with visualization in the geometric window. This will lead to a deductive justification for the equality of segments EM and MF. The tutor’s message (line 38) will help Marta to overcome obstacles related to the use of different labels for the figures in each environment in the execution phase. Marta proposes using measurement as an empirical argument (line 38), but she understands the necessity of proving as we can observe in the following paragraph. In the line 38, she proposes a naïve justification, but she makes the distinction between a proof and the ‘naïve justification’ proposed.

34. Marta: Bé, si mous P la variació és la mateixa [asenyala amb la mà segments a la pantalla per comparar la seva variació i escriu al full la següent igualtat errònia de raons: \( \frac{AP}{PD} = \frac{AB}{PE} \)].

35. Tutor: Fes servir les mateixes lletres que aquí [diferent labels in different environments, this may be due to a lack of confidence with the tool rename]

36. Marta: [Rectifica la igualtat anterior de raons i escriu les igualtats \( \frac{BM}{EM} = \frac{BA}{PE} \) i \( \frac{CM}{FM} = \frac{CA}{PF} \)]. I com que EM = MF… (Figure 4.13)

37. Tutor: No ho diu. Això ho has de demostrar.

38. Marta: Però com ho demostres si totes les lletres són diferents? Puc agafar nombres d’aquí per comprovar-ho [use of GGB measure tools]
In this case, the tutor guides the discussion with a cognitive message of level two (line 37) aimed at fostering deductive reasoning. She finally proves the initial conjecture about the equality of the segments EM and MF with the help of a cognitive message of level two (line 39).

For the second question, she encounters difficulties in constructing a “robust diagram”, which keeps its geometric properties when dragging the free elements. She tries to trisect the segment BC using dragging for adjusting. She uses the heuristic strategy of considering the problem solved (trisection of the segment BC), but she has technical difficulties. The acquisition degree of instrumentation, for this problem is low. She has difficulties with: a) applying the tool parallel line through a point to a line (she applies the tool to two segments), b) reporting distances to construct a robust figure (she does not consider the tool circle given the center and the radius), c) round-off error, d) hide and show elements. Marta does not use the dragging option to validate her conjectures in other triangles, but this may be due to the fact that the triangle of the statement is represented on a grid. She uses the drag tool to: a) to report distances, b) to test a conjecture, but she does not use the drag tool to modify the constructed figure (she does not drag the vertices of the triangle). We observe that she uses the dragging tool in the familiarization and analysis phases. In the exploration phase, she reasons on the static figure. Marta uses measure tools to explore relationships and she needs visual evidence to support her reasoning. She uses GGB in a reactive way (Hollebrands, 2007). She views what is presented on the screen and then perform another action.

During the exploration phase, Marta identifies that the point P is the centroid of the triangle, but the exploration phase does not lead to the construction of a proof of this fact. In this case, GGB helps Marta to find a conjecture but does not lead her to find a proof. The tutor plays the role of guiding the process with two conceptual messages of levels one and two. Marta finds out with the help of the tutor and measurement tools the property of the centroid (namely, that G trisects the 3 medians), which will suggest to her an empirical justification for the second question based on applying Thales property.

b) Marta’s resolution process of the median problem

For the resolution of the median problem, Marta follows a strategy based on justifying the equality of heights of both triangles, extracting two congruent triangles from the initial configuration. She does not mention the formula of the triangle’s area, but she uses it implicitly to justify the equality of areas considering that the triangles have a common base and equal correspondent heights.

Marta does not try to obtain concrete values neither for the areas of both triangles nor for the heights. We conjecture that the use of GGB helps Marta to understand the concept of quantifier in the statement of the problem: “P is any point on the median”. She had difficulties with this concept in the familiarization phases of the previous problems. It also may be due to the fact that there are no figures on the statement of this problem. Nevertheless, she considers a particular triangle ABC (isosceles triangle). Marta does not validate the conjectures in other triangles, but her deductions are not based on special properties of isosceles triangles. She understands the necessity of proof and that the exploration trough dragging is not sufficient to guarantee the truth of the observations made.
At the beginning of the planning-execution phase, Marta loses track of her initial solving strategy (Figure 4.16). She constructs the heights of the triangles, but she loses track of the resolution strategy, and she does not consider the common base AP of the triangles to construct the altitudes. Instead, she constructs an interior height of each triangle correctly, and she drags the point P along the median observing the figure on the geometric window (Figure 4.16). This may be due to the fact that she is not accustomed to working with heights that are not within the base of the triangle, but finally the use of GGB and tutor’s orchestration help Marta to overcome the obstacles encountered (visualization of the exterior height, selection of the equal heights).

8. Marta observes the figure [figure 4.17]
10. Marta: Tres…han de ser altures a la base comuna
11. Marta: [She constructs the perpendicular lines through B and C to the common base AM and she drags the point P along the median observing the geometric window]
12. Marta: Ostres, mira on és l’altura!
13. [Marta tries to construct the feet, H and F, of the altitude with the tool intersection of objects applied to the segment AP and the line e, exterior altitude. The software does not recognize the action as the segment and the line do not have any intersection point.]
14. Tutor: Has de considerar la recta (AP) per obtenir el punt d’intersecció amb la perpendicular
15. [Marta constructs the line AP and she applies the tool intersection to the lines AP and e, to obtain the feet H of the altitude. She obtains then the point F; she applies the tool distance to obtain CE and BF (Figure 4.18)]
16. [Marta drags the point P along the median (2 seconds) and she observes the equality of heights]

In the execution phase, for proving that the heights have always the same length, she reasons on the static figure. Marta extracts two congruent right-angled triangles from the configuration (operative apprehension) with the help of the tutor, and she justifies with a figural inference (Richard, 2004a) that both triangles are congruent. She applies congruence criteria of triangles, but she has conceptual difficulties with the criteria ASA.

20. Marta: Un altre cop la relació... entre FC i...Tinc un triangle aquí [AFC] i un triangle aquí... [ABH] (Figure 4.18)
21. Tutor: tracta de considerar altres triangles interiors
22. Marta: Un moment,... clar com que són paral·leles l’angle és el mateix [angles <C and <B]
23. Marta: Aquests triangles tenen un angle igual i un costat igual... són iguals
Marta considers the argumentation to be sufficient, and she finishes the solving process justifying with paper and pencil the equality of heights (Figure 4.17). She applies the congruence criteria to the triangles, but she does not consider angles contained by equal sides: \[ \triangle BMFH \equiv \triangle BMF \] (the equal sides are the hypotenuse of both triangles), otherwise the triangles that have equal angles and a side of same length are not necessarily congruent. In this case, Marta does not mention that the sides which have same length are the hypotenuses of both right-angled triangles.

Marta constructs figures based in geometric properties, but she tends to use measure tools more often than other students. Another relevant fact is that she uses discrete dragging (Olivero, 2002) to test the conjectures and in the exploration phase. She tries to reason on the static figure and creates particular cases by dragging. She also understands the necessity of proving (lines 4-7).

4. Marta: Tenen la mateixa àrea. Bé, ara he de justificar perquè…
5. [Marta drags the point P along the median to obtain particular cases P=M, P=A, P midpoint of the median (visual adjusting) observing the area in the algebraic window (2 seconds).]
6. [Marta observes the static figure and drags again the point P along the median (2 seconds)]
7. Marta: és com abans…l’altura seria…

A relevant aspect is that Marta considers changing the initial triangle ABC. In the evaluation phase, she reacts to the tutor’s message (lines 27-30) by dragging the vertices B and C. Marta observes that the median splits the triangle into two equal triangles (equivalent dissection rules), but it depends on the triangle. She drags the vertices B and
C to observe this fact. For the first time, she considers dragging the initial triangle (Figure 4.20).

27. Tutor: Podries resoldre aquest problema utilitzant una estrategia diferent?
28. Marta: no. No sé com ho podria resoldre…
29. Tutor: podries utilitzar una estrategia basada en descomposició d’àrees?
30. Marta: és que aquesta línia [median] parteix el triangle per la meitat [two congruent triangles] però clar, això depèn del triangle… [She drags the vertices B and C to obtain a general triangle; the previous triangle looked like an equilateral triangle in which both inner triangles were congruent]

Finally, we observe again that some technical obstacles are related to already existent cognitive obstacles (Drijvers, 2002). For example, the technical difficulty with the tool intersection (line 28) is related to the difficulty in visualizing the exterior height of the triangle. As already demonstrated by several researchers, students are not accustomed to working with heights that are not within the base of the triangle.

c) Marta’s resolution process of the quadrilateral problem
For the resolution of the quadrilateral problem, Marta follows the strategy based on justifying that the sum of the triangles’ areas is constant, and thus the area of the quadrilateral is constant. Marta splits the quadrilateral into two triangles to obtain its area as the addition of triangles’ areas, and she conjectures that these constants are equal. The tutor’s orchestration helps Marta to justify that the segment MN is half of the segment AC, and thus the areas are equal, but she has difficulties in visualizing this relation.

Marta justifies deductively the ratio 1:2 of the segments MN and BC by applying Thales’ theorem. She considers the ratio of homolog sides of similar triangles BMN and BAC.
In the resolution of the problem, Marta understands the notion of quantifier “P any point of the segment AC”. For instance, Marta considers different positions of the point P (Figures 4.21, 4.22, 4.23); this is a relevant difference with respect the first paper-and-pencil problem, and with the second problem, in which the tutor has to suggest that Marta drag the point P along the median. In the familiarization phase, Marta realizes the fact that by drawing the point P she determines a concrete position of the point.
In the exploration phase, Marta tries to recognize geometric properties, but she has represented a triangle which is close to an isosceles triangle, and through perceptual approach she conjectures that MN and the perpendicular line from B to the side AC
intersect at the midpoint, but she does not appreciate that this property holds only for isosceles triangles (Figure 4.21).

Figure 4.21: She draws MN and the perpendicular line to MN through B (height of the triangle ABC through B)

Nevertheless, the strategy based on splitting the quadrilateral into two triangles to obtain its area and justifying that the heights of both triangles, AMN and MNP, have the same length is adequate. Marta abandons her strategy based on obtaining the area of the quadrilateral, as she does not trust the previous property. She reads the statement of the problem and starts again the exploration phase. She tries to justify that the heights (from M and N) of the triangles AMP and PNC have the same length. Marta validates the conjecture considering particular cases (right-angled triangles).

10. Marta: Jo que sé...si el triangle és així [right-angled triangle], clar l’altura també és igual aquí… [Figure 4.22]
11. Marta: És que jo crec que l’altura sempre és igual [triangles AMP and PNC]
12. Marta: …la recta MN és paral·le·la a la base [AC]. Sí, sí sempre ho és així
13. Tutor: Com justificaries que aquestes rectes són paral·leles?
   [Marta remains silent]
   […]

The tutor’s intervention marks the beginning of an execution phase: ‘Quina relació hi ha entre els segments determinats sobre les rectes tallades per paral·leles?’ (line 17 of the paragraph below). She tries to justify the equality of heights, but is not able to justify deductively the parallelism of the lines MN and AC. She does not consider the use of Thales theorem to justify this fact. Marta is used to applying the theorem when having the parallelism as hypothesis. She starts a short exploration, but abandons the search for a proof and accepts the truth of this result. The tutor tries to foster Marta’s deductive reasoning by giving a cognitive message of level 1 ‘Com justificaries que aquestes rectes són paral·leles?’ (line 13). As she has difficulties, the tutor gives Marta the following cognitive message of level 2: ‘Quina relació hi ha entre els segments determinats per les rectes tallades per paral·leles?’ (line 14).

Marta reacts to this message by starting again an exploration phase. She considers proportional segments and the use of Thales theorem. Marta tries to justify the parallelism (see paragraph below).

17. Tutor: Quina relació hi ha entre els segments determinats sobre les rectes tallades per paral·leles?
18. Marta: els segments són proporcional
19. Tutor: què vols dir?
20. Marta: O sigui això partit per això…
21. Marta: Bueno, suposo que de C a N partit de CB serà igual que de A a M partit per AB.
22 Tutor: Com podries justificar que la relació és un mig
23. Marta: Com ho sé que és un mig?
24. Tutor: recorta com estan definits els punts M i N [cognitive message of level one]
25. Marta: Ah, sí! Són els punts mitjos. Serien les rectes MN i AC paral·leles que els defineixen [segments]
26. Tutor: Quina relació hi ha entre segments paral·lels determinats per rectes paral·leles?
27. Marta: són iguals….les alçades… [She does not prove Thales reciprocal]

Finally, the tutor suggests to Marta the strategy based on justifying the equality of heights. Marta starts an exploration phase and tries to justify the equality of heights for the inner triangles of the quadrilateral, but she has difficulties in understanding that the heights are equal. We conjecture that this obstacle is related to the fact that she has difficulties in visualizing the height as the distance between parallel lines and in transforming the initial configuration to apply Thales theorem. Marta considers a particular right-angled triangle to visualize Thales configuration and thus to justify the equality of ratios (Figure 4.24), but she is aware of the fact that this is only a particular case.

![Figure 4.22: Particular point P (collinear heights)](image)

![Figure 4.23: Marta’s generalization (point P)](image)

28. Tutor: Com podries justificar que el triangle BMN també té la mateixa alçada
29. Marta: Perquè? Ah, sí, perquè fas així [Figure 4.22]
30. Marta: No, però mira això… [Figure 4.23] Clar, però aquí no és exacte...
31. Tutor: intenta justificar que l’alçada d’aquests triangle són iguals [BMN and MNP]
32. Marta: perquè són iguals?
33. Tutor: recorda que M i N són punts mitjos.
39. Marta: la resta entre una àrea [MPNB] i l’altre [triangles AMP and PNC] sempre serà constant. Els altres dos [both triangles AMP and PNC] ja he vist que és constant
40. Tutor: Però quina relació tenen?
41. Marta: Que la resta és constant. Això és una relació?
42. Tutor: Si, però quina constant?
43. Marta: si fas un triangle rectangle es veu. Clar és un exemple… [equality of height]

![Figure 4.24: Particular case, equality of heights: BM=MA](image)
Marta justifies the equality of heights (naïve justification) and conjectures a relation between the area of the quadrilateral and the addition of the areas of both triangles. We observe that she has difficulties in stating a conjecture about the relation between the areas of the figures. For instance, she states that the subtraction of both areas is constant as a possible relation, but she does not observe that the addition of both areas is constant. Finally, with the help of the tutor she justifies that both areas are equal, but she does not notice that the area of the quadrilateral is one half of the triangle ABC.

She reacts to the last message by trying to conjecture again a relation between both areas. She conjectures the equality of areas. She may have observed this visually in the particular cases, and she also may be influenced by the previous problems (equality of areas). The tutor’s orchestration has a key role in the last planning execution phase (line 48). The tutor suggests to Marta the relation $2\text{MN}=\text{AC}$, and Marta justifies it deductively by applying Thales theorem. She considers Thales configuration to compare homolog sides of similar triangles (line 51).

Figure 4.25: Marta’s drawing to illustrate that $2\text{MN}=\text{AC}$

47. Marta: És que jo diria que són iguals [Conjecture: equality of areas]
48. Tutor: quina relació hi ha d’haver entre els segments MN i AC? [Cognitive message of level two]
49. Marta: I no puc fer relació de això partit per això? O sigui aquesta relació és igual que aquesta?
50. Tutor: Com ho justificaries?
51. Marta: Perquè serien triangles semblants. Bé, és Thales això. Si sabem que això és la meitat d’això, això és la meitat d’això. [Figure 4.25]

Marta justifies the equality of areas using an algebraic strategy (Figure 4.26). She obtains the expressions for both areas and she compares these expressions. She obtains algebraic expressions for both areas using the geometric properties proved (equal heights, $2\text{MN}=\text{AC}$).
A relevant aspect of this resolution is that Marta does not consider that if both areas are equal, each area is half the area of the exterior triangle ABC. It may be due again to an algebraic-visual obstacle. She does not visualize the resolution of a linear equation: $2C = (ABC)$. Moreover, she does not consider that the areas are complementary; she does not need to obtain the area of the quadrilateral once she has obtained the area of both inner triangles.

### 4.2.1.2 Summary of the protocol: The case of Guillem

#### a) Guillem’s resolution process of the scaled triangles problem

In the familiarization phase, Guillem tries first to reproduce the figure of the problem’s statement with GeoGebra. He states that he does not have enough information to construct the figure (construction of the points P, E and F) as he considers that he has to reproduce the concrete figure of the problem’s statement. This misconception may be due to the fact that the figure of the statement is represented on a grid, and students interpret that the triangle ABC is defined with the lengths of its segments and the point P is a specific point on the median. However, in Guillem’s case, it is also due to a technical obstacle, as we will see in the following paragraph.

1. Guillem: Ho faig com aquí [Figure 4.27]. Agafa 6 unitats [he constructs a segment BC of length 6 units using the grid. He is trying to reproduce the figure that appears in the statement of the problem.]
2. Guillem: P és un punt qualsevol de la mitjana … [he reads the statement of the problem]
3. Guillem: No ho puc dibuixar perquè no tinc les dades, P varia. Hauria de fer les paral·leles que es puguin moure.
4. Guillem: Hauria de moure P [guided dragging to trisect the segment BC]…. [He remains in silence observing the triangle on the screen]
The tutor suggests that Guillem read again the statement of the problem and gives him a conceptual message of level two (line 8) suggesting the construction of point P.

7. [Guillem tries to construct the parallel lines before constructing the point P. He clicks on two segments instead of clicking on the segment or line and a point (technical obstacle)]
8. Tutor: per utilitzar l’eina paral·lela a una recta per un punt has de definir el punt. [Contextual message of level two]
9. Guillem: Però, si poso el punt, quedàrà fix [technical obstacle]
10. Tutor: No, un punt qualsevol de la mitjana es pot desplaçar sobre la mitjana. [Contextual message of level three]
11. [Guillem constructs with GeoGebra the point P on the median and the parallel lines trough P to the sides of the triangle.]

Guillem has technical difficulties with the concept of dependent objects (line 9). The tutor’s contextual message (line 10) helps Guillem to overcome this technical obstacle. Guillem constructs the point P and the parallel lines through P to the sides (AB) and (AC) of the triangle. The reproduction of the figure with GeoGebra helps Guillem to visualize and to understand the problem. He starts an analysis/exploration phase by trying to state a conjecture He uses the drag tool to validate the conjecture, as we can see in the following paragraph.

16. Guillem: He de veure la relació entre EM i MF.
17. Guillem: Són iguals… [He observes on the geometric window that the segments have the same length]
18. Guillem: Ara vario P per veure que sempre és igual….
19. Guillem: Sí, són iguals. Ara s’ha de justificar

Guillem understands that he has to justify the conjecture (necessity of proof), and he uses the drag tool to validate the conjecture, as we can see in the above paragraph. Guillem obtains expressions in terms of EM and FM using Thales theorem (he uses the label D, instead of M in the GGB construction), but he is not able to deduce the equality of segments (EM and MF). He considers then a strategy based on isolating EM and MF as a last resort, but he gets lost. The tutor message reminds Guillem of the strategy based on applying Thales theorem twice, but comparing the common side, AD, of both triangles (ABD and ADC). Guillem discerns the inner triangles ABD, EPD and PFD, ADC to apply Thales theorem, but he does not extract the similar triangle EPF. As we will see in the following paragraph, he also has difficulties in visualizing the equivalence of ratios DE: DB= DF: FC.

20. [Guillem applies Thales theorem and he obtains: $\frac{BM}{EM} = \frac{BA}{PE}$ and $\frac{CM}{FM} = \frac{CA}{PF}$ (M=D in Figure 4.27).]
21. Guillem: he de demostrar que EM és igual a MF….
22. Guillem: Aïllo EM i MF. Llavors amb el resultat que quedi, ja veuré…
23. [Guillem isolates EM and MF in the previous expressions and observes the result in silence. There is no feedback during four minutes]
24. Tutor: Tracta de considerer el costat comú als dos triangles [Cognitive message of level two]
25. [Guillem applies Thales theorem again considering the common side AM of both triangles ABM and AMC. Guillem obtains the relations: $\frac{BM}{EM} = \frac{AM}{PM}$ and $\frac{MC}{MF} = \frac{AM}{PM}$ ]
The tutor’s message (line 24) will help Guillem to prove that EM and FM are equal. The organization of the reasoning process is:

Dynamic Figure  \[ \frac{BM}{EM} = \frac{AM}{PM} \]

\[ \rightarrow \frac{BM}{EM} = \frac{MC}{MF} \quad \text{Midpo int[} BC]\] \[ \rightarrow \text{EM}= MC \]

Dynamic Figure  \[ \frac{MC}{MF} = \frac{AM}{PM} \]

During the execution phase, he uses labels that appear in the figure of the statement of the problem (M instead of D). This may be due to technical difficulties with the option rename objects. Guillem deduces the expression \( \frac{BM}{EM} = \frac{MC}{MF} \) using Thales property twice and using that M is the midpoint of BC, he deduces that EM = MF (Figure 4.28). We code his reasoning process with a (+) score in the deductive competence. Moreover, we interpret from Guillem’s worksheet that he understands the concept of parameter (MC=MB= a, Figure 4.28). There is a shift in the structural competence with respect to the previous problem.
For the second question of the problem, Guillem considers again the figure on the screen. We observe the heuristic strategy of considering the problem solved (he drags the point P along the median to obtain the trisection of the segment AC). Although it is possible to apply Thales theorem, Guillem does not consider this strategy immediately, as in the previous question. He tries to find geometric properties of the point P in the figure, that suggest to him a solving strategy.

Guillem conjectures that the point P is the intersection of the medians of the triangle ABC, but he does not know how to justify this fact. As Hoyles and Healy (1999) state, exploration of geometrical concepts using DGS helps the students to define and identify properties, but not necessarily lead to the construction of a proof.

Guillem remains silent for a while observing the figure on the screen (Figure 4.29).

With the help of the tutor, he finds out that the centroid is a point of trisection of the median, \(3 \cdot ED = BD\) (he had technical difficulties related to the round-off error). The technical difficulties in interpreting the round-off error are also related to the fact that he does not validate the properties of the centroid in other triangles. He uses this property as a key idea to justify that the points E and F trisect the base of the triangle, but he does not validate the conjecture in other triangles (dragging option). As mentioned in the previous section, it takes time before students apply the drag mode spontaneously. First, he begins to drag his diagrams locally and partially. In this case, it also may be due to the fact that
the tutor gives him a message about the relation between these segments (3ED=BD). Nevertheless, he discovers again this property with the use of measure tools applied to the segments.

Guillem justifies with paper and pencil his conjecture considering the properties of the centroid. He makes a graphic representation without coordinate axes (Figure 4.31) and tries to justify the conjecture. Guillem bases his deductions on the fact, justified empirically, that the medians trisect each other at the centroid. He uses first the fact that the three medians of any triangle all pass through one point as an accepted result. In this case, the use of GeoGebra suggests to Guillem the strategy based on using the centroid properties. He introduces another median (operative apprehension) and applies Thales theorem to obtain the trisection of the segment AC considering a parallel line through the centroid to the side (BC) (Figure 4.30). He states (line 36)-“La distància de P [centroid] a D [M] és un terç de la distància total, per tant d’aquí a aquí també [the segments defined in the side BC].” Nevertheless, Guillem does not construct with GeoGebra the parallel lines. He avoids overcrowding the figure, as we can see in the paragraph below. This could be due to a lack of confidence with GeoGebra.

28. [Guillem drags the point P to obtain visually the trisection of the segment]
29. Guillem: Ara s’ha de justificar
30. [Guillem constructs the other median and observes that the point P belongs to the median]
31. Guillem: I si fes una altra paral·lela, aquests tres segur que també serien iguals. Però ja hi han masses coses en el dibuix…
32. Guillem: no sé…
33. Tutor: Tracta de recordar les propietats del baricentre que és el punt d’intersecció de les mitjanes
34. Guillem: Les tres àrees són iguals [EAB, EAC and ECA].
35. Tutor: Quina és la relació entre els segments BD i ED? [Conceptual message of level 2]
36. Guillem: La distància de P (centroid E) a D (M) és un terç de la distància total [median], per tant d’aquí a aquí també [the segments defined in the side BC]. [He does not validate the property of the centroid in other triangles]

Figure 4.30: solving strategy based on the centroid properties and Thales theorem (E stands for P, D for M, G for E and H for F)
During the resolution process of this problem, Guillem overcomes technical difficulties related to the dependency relationships of the elements of the problem. The use of GGB and the tutor’s orchestration facilitate Guillem’s understanding of the problem. During the exploration phase, Guillem identifies Thales property by using the dragging option combined with visualization in the geometric window. This will lead to a proof for the equality of segments EM and MF (cognitive unity). Guillem proves the initial conjecture about the equality of the segments EM and MF with the help of the tutor.

For the second question, Guillem encounters difficulties in constructing a “robust diagram”, which keeps its properties in the drag mode. He trisects the segment BC using dragging for adjusting, and he uses the heuristic strategy of considering the problem solved (trisection of the segment BC), but he has technical difficulties. The acquisition degree of instrumentation for this problem is low. He has difficulties with: a) applying the tool parallel line (he applies it to 2 segments instead of a segment or line and a point), b) reporting distances to construct a robust figure (he does not consider the tool circle given the center and the radius), c) interpreting round-off error obstacles, d) applying the tool hide and show elements (this obstacle leads Guillem to avoid the construction of auxiliary elements. Guillem does not use the dragging option to validate their conjectures in other triangles, but it may be due to the fact that the triangle of the statement is
represented on a grid. He uses the drag tool to: a) to report distances, b) to test a conjecture, but he does not use the drag tool to modify the constructed figure (he does not drag the vertices of the triangle). Guillem uses the drag tool to validate that the geometric property is maintained. For example, he drags the point P along the median AM to check the equality (linked dragging). We observe that he uses this tool in the familiarization and analysis phases. Guillem uses GGB proactively. He appears to have a plan in mind about how to use GGB. For example, he decides to define auxiliary elements with certain expectations of what he wants to observe.

In this case, GGB helps Guillem to find a conjecture, but does not lead him to find a proof. The tutor plays the role of guiding the process with two conceptual messages of levels one and two. Guillem finds out with the help of the tutor and measurement tools the property of the centroid (namely, that G trisects the 3 medians) that will suggest a justification for the second question based on applying Thales property.

**b) Guillem’s resolution process of the median problem**

For the resolution of the median problem, Guillem follows a strategy based on justifying the equality of heights of both triangles, extracting two congruent triangles from the initial configuration. Guillem does not mention the formula of the triangle’s area, but he uses it implicitly when he deduces that the equality of areas is equivalent to the equality of heights (simple reduction lemma). We code it with a (+) score into the instrumental competence.

\[(APC) = (APB) \quad \text{CommonBaseAP} \quad \text{height}_B = \text{height}_C.\]

The use of GeoGebra has a key role in this deduction. The geometric and algebraic windows combined with the continuous dragging (Olivero, 2002) provide Guillem with perceptual evidence that the area and the length of the base are equal for both triangles. Guillem deduces that if the areas are equal the heights have to be equal (Figure 4.32).

6. [Guillem observes on the algebraic window that the areas are equal]
7. [Guillem drags continuously the point P along the median while looking at the variations on the algebraic and on the geometric window (9 seconds)]
8. Guillem: [He observes again the static figure] Doncs perqué deuen de tenir la mateixa altura, suposo. He de buscar les altures [construction] per saber que són iguals.
9. Guillem: [He drags continuously the point P along the median while looking at the variations on the algebraic and on the geometric window (6 seconds)] Ah, sí. La base [common base AP] canvia però canvia pels dos i les altures són iguals
10. Guillem: Ara hauria de buscar... les altures per saber que són iguals. I ja tinc l’explicació.

Guillem does not try to obtain concrete values, as he did in the root problem, neither for the areas of both triangles nor for the heights. We conjecture that the use of GeoGebra
helps Guillem to understand the concept of quantifier in the statement of the problem: “P is any point on the median”. It also may be due to the fact that there are no figures on the statement of the problem. Nevertheless, Guillem considers a particular triangle ABC (isosceles triangle). Guillem states:

2. Guillem: Com que no fa referència a mides puc fer un triangle rectangle.
3. Tutor: Sí, però el raonament que facis s’ha de generalitzar després per a un triangle qualsevol.
4. [Guillem constructs with the tool polygon, using the grid, the points A(0,0), B(0,6) and C(3,4) which forma n isosceles triangle]

The tutor reminds Guillem that his reasoning has to be valid for any triangle (line 3). Nevertheless, Guillem does not react to this message. He constructs an isosceles triangle. He does not validate the conjectures in other triangles, but his deductions are not based on special properties of isosceles triangles.

Guillem understands the necessity of proof and realizes that the exploration trough dragging is not sufficient to guarantee the truth of the observations made, as we can see in the following statement (line 23): ‘La base sempre serà la mateixa. Llavors, he de demostrar perquè les altures són iguals... [He remains silent observing the static figure]’.

We conjecture that, at the beginning of the exploration phase, the immediate perceptual approach of the triangles (Figure 4.32) is an obstacle for the geometric interpretation. For example, he validates visually the conjecture C₃ (P is the midpoint of the segment defined by the intersection of the altitude with the sides of the triangle [Figure 4.33]). Guillem encounters difficulties with the visualization of the exterior heights, as we can see in the paragraph below.

12. [Guillem drags the point P along the median (12 seconds) and he states a conjecture] He de demostrar que P està en el punt mig de les interseccions de la perpendicular amb els costats [AC and AB (Figure 4.33)]
13. Tutor: Recorda la definició d’altura d’un triangle. Com ha de ser l’ altura de cada triangle?
14. Guillem: He de fer dos altures?
15. Tutor: Una per cada triangle
16. [Guillem constructs the altitudes using the tool perpendicular line to a given line through a point.]
17. [Guillem tries to construct the feet, H and F, of the altitude with the tool intersection of objects applied to the segment AP and the line e, exterior altitude. The software does not recognize the action as the segment and the line do not have any intersection point.
18. Tutor: Has de considerar la recta (AP) per obtenir el punt d’intersecció amb la perpendicular
19. [Guillem constructs the line AP and he applies the tool intersection to the lines AP and e, to obtain the feet H of the altitude. He obtains then the point F and applies the tool distance to obtain CE and BF]

Figure 4.33: False conjecture (C₃) validated visually with the drag tool: P is the midpoint of the segments on perpendicular line through P (Guillem)
Guillem tends to base his constructions on geometric properties. He states, for example, that he uses the tool polygon to construct a triangle because using the tool segment the figure may be no longer a triangle when dragging the initial points. But he tends to avoid long constructions (construction of the feet of the altitudes for example) and the use of measure tools. We have already observed this behaviour in the previous problem; he stated ‘ja hi ha masses coses en el dibuix’. He tests his conjectures via continuous dragging and perceptual approach. For example, he observes that the heights remain invariant when dragging the point P along the median. He poses and tests conjectures with the help of GeoGebra \((C_1, C_2, C_3)\), but he understands the need of proving.

In the execution phase, for proving that the heights have always the same length, Guillem reasons on the static figure. Guillem extracts congruent right-angled triangles from the configuration (operative apprehension) and justifies deductively that both triangles are congruent using trigonometry.

23. Guillem: La base sempre serà la mateixa. Llavors, he de demostrar perquè les altures són iguals... [He remains silent observing the static figure].
24. Guillem: No sé... La hipotenusa d’aquests dos triangles és la mateixa [BHM and MFC] perquè M és el punt mig. El que faltaría és... [HM = MF] (Figure 4.34)
25. Guillem: Ja està. Tenen un angle igual i un costat igual...
26. Tutor: És suficient?
27. Guillem: Clar. Si tenen la hipotenusa i un angle el cosinus sempre serà igual...

![Figure 4.34: Guillem’s worksheet](image)

Finally, we show that some technical obstacles are related to already existent cognitive obstacles (Drijvers, 2002). For example the technical difficulty with the tool intersection (lines 17, 18) is related to the difficulty in visualizing the exterior height of the triangle. As demonstrated by many researchers, students are not accustomed to working with heights that are not within the base of the triangle. Technical obstacles are also opportunities for developing mathematical knowledge. For example, the technical difficulty encountered by Guillem with the dragging tool allows him to understand the motion dependency in terms of logical dependency within the geometrical context (when dragging the vertex A along a parallel line to the median, the point P is indirectly dragged along the median).
c) Guillem’s resolution process of the quadrilateral problem

For the resolution of the quadrilateral problem, Guillem’s strategy is based on justifying that the sum of the triangles’ areas is constant and thus the area of the quadrilateral is constant. Guillem splits the quadrilateral into two triangles to obtain its area as the addition of triangles’ areas, and he conjectures that these constants are equal. Through a *figural inference* (Richard, 2004a) Guillem states that the segment MN has to be half of the segment AC if the areas are equal, but he has difficulties in justifying this relation. Guillem’s *figural inference* has the following tertiary structure:

\[
(BMN) + (MNP) = (AMP) + (NPC)
\]

\[
MN = \frac{1}{2} AC
\]

The tutor has a key role in helping Guillem to justify deductively the ratio 1:2 of the segments MN and BC applying Thales theorem. Guillem considers the ratio of segments by applying Thales theorem.

In the resolution of this problem, for the first time, he represents a non-particular triangle, with an oblique position on the worksheet (Figure 4.36) and he considers other triangles to validate conjectures, to find counterexamples, etc. Moreover, Guillem understands the the notion of quantifier “P any point of the segment AC”. He understands the logical structure of the problem, as we can see in the paragraph below:

3. Guillem: Jo crec… que sempre tenen la mateixa àrea perquè aquí les altures no varien…Saps que vull dir…els dos triangles…Bueno, no sé… [areas of the triangles AMP and PNC (Figure 4.36)]

5. [Guillem observes the figure in silence]

6. Guillem: Si sé que l’àrea de la suma dels dos no varia vol dir que l’altre [BNMP] no varia per molt que tingui el punt P. L’unic canvi és que un perd base [AMP] i l’altre en guanya [NPC] però les altures en principi han de ser fixes

7. Guillem: he de buscar primer que… l’altura… Hi ha d’haver algo que em digui que l’altura sempre és igual en els dos triangles
Guillem starts the exploration phase by trying to find invariants. He infers that the sum of the areas of the triangles AMP and NMP is constant. This inference is a figural inference of binary structure:

(Figure 4.36) ‘*quan un perd base l’altre en guanya i les àlçades són igual (perceptual apprehension)*’  \[ \text{(AMP)+(PNC)} = \text{constant} \]

Guillem does not use the fact that the sum of areas does not depend on the position of the point P to conjecture the area through the consideration of particular positions of the point P. He conjectures that both areas are constant, but he does not conjecture that each area (the addition of both triangles’ areas and the quadrilateral’ area) is half the area of the triangle. This may be due to a visualization difficulty, as we will see later. As he mentally drags the point P along the side AC, we conjecture that he does not have in mind the property of the median (it splits the triangle into two triangles of equal area) or that he does not visualize the degenerate case (P=B or P=C).

Guillem’s strategy is based on justifying that the heights (from M and N) of the triangles AMP and PNC have the same length. Guillem’s intervention (line 13) marks the beginning of an execution phase. He tries to justify the equality of heights. Guillem is not certain about the conjecture, as we can see in the table below (Table 4.4).

<table>
<thead>
<tr>
<th>Dialogues and construction steps</th>
<th>Strategy</th>
<th>Argumentation</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>8. Guillem: Jo crec que l’altura sempre és... Es poden unir els punts mitjons amb una recta paral·lela a aquesta [AC] i això em diria que l’altura és igual...</td>
<td>He tries to validate the conjecture</td>
<td>Semantic inference</td>
<td>Construction of an auxiliary element: segment MN</td>
</tr>
<tr>
<td>9. Guillem: …Però, clar aquí sense el GeoGebra no ho sabes…a lo millor el punt mig és una mica més avall...</td>
<td>Semantic inference</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Tutor: Què vols dir amb això?</td>
<td>Message of level 1 (on the spot orchestration)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Guillem: Tu pots imaginar que una paral·lela pugui passar per M i N però no saps segur si el punt mig està a la paral·lela o una mica més avall. Com que no treballem amb ordinador...</td>
<td>He is not certain that the conjecture is valid GeoGebra dependence: he is not aware that the GGB plane has a finite number of points</td>
<td>Necessity of proof</td>
<td></td>
</tr>
</tbody>
</table>
12. [Guillem draws many triangles to check the property: the line through the midpoint M, parallel to BC join the midpoint N] He is not sure that this result is true for all possible variations of the triangle. He considers other triangles and a singular case to validate the conjecture in a larger number of examples (he considers also a degenerate case of triangle) necessity of proof

Figure 4.37: validation of the property (Guillem)

13. Tutor: Què fas?
15. Tutor: com podries justificar que l’altura és la mateixa?
16. Guillem: Clar és que és això…Ah! Vale, si faig una recta paral·lela per la meitat això quedará partit per la meitat. Però no sé si una paral·lela pel punt mig tala pel punt mig.

Table 4.4: Guillem’s argumentations (equality of heights)

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>12.</td>
<td>Guillem draws many triangles to check the property: the line through the midpoint M, parallel to BC join the midpoint N</td>
<td>He is not sure that this result is true for all possible variations of the triangle. He considers other triangles and a singular case to validate the conjecture in a larger number of examples (he considers also a degenerate case of triangle) necessity of proof</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>Tutor: Què fas?</td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>14.</td>
<td>Guillem: Veure-lo. Jo crec que sempre és lo mateix</td>
<td>Validation of the conjecture</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>Tutor: com podries justificar que l’altura és la mateixa?</td>
<td>Cognitive message of level 1 (he is already trying to consider the heights)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Objective of the message: Modify the epistemic value of the heights equality</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>16.</td>
<td>Guillem: Clar és que és això…Ah! Vale, si faig una recta paral·lela per la meitat això quedará partit per la meitat. Però no sé si una paral·lela pel punt mig tala pel punt mig.</td>
<td>Justifying the equality of heights Difficulty in proving Thales reciprocal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Semantic inference (he does not mention Thales property, or the uniqueness of the midpoint)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Perceptual approach</td>
</tr>
</tbody>
</table>

Guillem states ‘Tu pots imaginar que una recta paral·lela pugui passar per M i N però no saps segur si el punt mig està a la paral·lela o una mica més avall’ (Table 4.4, line 11). Guillem is aware of the lack of precision of the paper-and-pencil drawing (lines 8-11) and the need of reasoning on the base of accepted properties. He makes the distinction between figure and drawing, as we can observe in the graphic action consisting in drawing different triangles, and even a singular case, to check the parallelism (Table 4.4, Figure 4.37). As stated by Laborde and Capponi (1994), a drawing can not characterize the geometric object. Guillem is not able to justify deductively the parallelism of the line through the midpoints. He does not consider the use of Thales theorem to justify this fact, as he expects to use it with the parallelism as hypothesis. Guillem considers then justifying the equality of angles, but he is aware of the equivalence of both properties. In fact this is related to Euclid’s fifth postulate. He abandons the search for a proof and he tries to accept the local result, as we can see in the paragraph below.

17. Guillem: Puc posar això? Com que sempre que faig una paral·lela de AC, passa per M i N vol dir que l’altura sempre serà la mateixa pels dos triangles
18. Tutor: Com justificaries que aquestes rectes són paral·leles?
19. [He remaing silent]
20. Guillem: Ah! Vale… Aquests angles…però són iguals si les rectes són paral·leles…
21. Guillem: no, és igual, l’angle ha de ser el mateix en el cas que siguin paral·leles. Podries mesurar… apreciar quan un angle és recte fent perpendiculars per justificar que com les dos són perpendiculars a aquestes, han de ser paral·leles.
22. Tutor: També ho hauries de justificar. Quina relació hi ha entre segments determinats sobre les rectes tallades per paral·leles?

The tutor tries to foster Guillem’s deductive reasoning by giving him a cognitive message of level one: ‘Com justificaries que aquestes rectes són paral·leles’ (line 18). As Guillem has difficulties, the tutor gives him the following cognitive message of level two: ‘Quina relació hi ha entre els segments determinats per les rectes tallades per paral·leles?’ (line 22). Guillem reacts to the previous message by trying to justify deductively the parallelism of both lines. Guillem is near a deductive proof, because by drawing a parallel line through the midpoint M he obtains a line r which intersects the segment BC in a point N’. By applying Thales theorem this point is the midpoint N. Thus the line MN is the line r (uniqueness of lines by two points), which is parallel to the line (AC). Finally, he accepts the truth of the conjecture and starts an analysis phase (line 36) and decides to apply Thales property again to justify the equality of heights for the inner triangles of the quadrilateral, as we can see in the paragraph below.

30. Guillem: Ara sé que la suma de les àrees no varia [(AMP) + (PNC) constant]. Clar,…però de la mateixa manera… si traço una paral·lela sé que d’aquí a aquí hi ha la mateixa distància
31. Guillem: L’altura del triangle serà la mateixa. Faig una paral·lela per B a MN [Figure 4.38]
32. Guillem: l’altura d’aquest triangle [BMN] serà la mateixa que la d’aquest [MNP]
33. Tutor: Com ho podrís justificar?
34. Guillem: perquè és el punt mig. Com que AM és igual a MB…faig una paral·lela per B a MN llavors l’altura ha de ser la mateixa
35. Guillem: les paral·leles divideixen l’altura total en dos parts iguals. Aquest punt seria…A’ [Figure 4.38]

Figure 4.38: Guillem’s triangle, he draws segment BB’ and the segment B’A’

Guillem starts an analysis/exploration phase. He conjectures the equality of areas. He may have observed this visually on the particular cases, and also he may be influenced by the previous problems (equality of areas). The tutor’s orchestration has a key role in the last planning execution phase. Guillem’s infers the relation 2MN=AC (figural inference) and the tutor helps Guillem to justify it deductively by considering Thales theorem. Finally, Guillem considers Thales configuration to compare homolog sides of similar triangles.

41. Guillem: Jo crec que hauria de dir, bé imagino, que MN és un mig, és la meitat de AC.
42. Tutor: Què vols dir?
43. Guillem: Clar perquè, aquí l’àrea seria x per MN i llavors multipliques per 2 [Figure 4.38]. Perquè el quadrilàter tingui la mateixa àrea que els dos triangles…
44. Tutor: com saps que tenen la mateixa àrea?
45. Guillem: Una relació que seria possible és que el quadrilàter tingui la mateixa àrea que els dos triangles
46. Guillem: Clar i perquè això sigui, la base MN ha de ser la meitat de AC.
47. Tutor: com podries justificar aquesta relació? [AC=2MN]
48. Guillem: no sé...
49. Tutor: tracta de trobar relacions entre els costats d’aquests triangles [BMN and BAC]

Finally Guillem, with the help of the tutor (cognitive message of level 2), applies Thales theorem to deduce that AC=2MN. Guillem’s strategy has the following structure:

\[
\begin{align*}
S_1 &= (BMNP) = \text{constant } C \\
S_2 &= (AMP) + (PNC) = \text{constant } C' \\
\text{Conjecture: } S_1 &= S_2 \\
\text{Figural inference: } S_1 &= S_2 \iff AC=2MN \text{ (reconfigurative operative apprehension)} \\
\text{Thales theorem: } AC &= 2MN \text{ thus the areas are equal.}
\end{align*}
\]

A relevant aspect of this resolution is that Guillem does not consider that if both areas are equal, each area is half the area or the exterior triangle ABC. It may be due again to an algebraic-visual obstacle. He does not visualize the resolution of a linear equation: \(2C = (ABC)\). Moreover, he does not consider that the areas are complementary; he does not need to obtain the area of the quadrilateral once he has obtained the area of both inner triangles.

**4.2.1.3 Summary of the protocol: The case of Aleix**

**a) Aleix’s resolution process of the scaled triangles problem**

For the resolution of this problem, Aleix uses a strategy based on identifying similar triangles. He identifies the inside triangle as a scaled copy of the outside triangle and compares pairs of corresponding sides to obtain the factor scale.

In the familiarization phase, Aleix reads the statement of the problem and constructs with GeoGebra the figure associated to the statement of the problem. He uses the tool polygon and clicks directly on the geometric window to define the vertices. A relevant aspect of the construction is that Aleix does not pay attention to the grid. He constructs a triangle, in horizontal position, which is close to the figure of the statement, but he does not pay attention to the vertices coordinates. He only tries to use the same labels for the vertices (modifying it with the option properties of an object). He understands that the statement holds for any triangle. The following fragment shows that he also understands that the point P is any point of the median.

4. Aleix: els punts E i F són qualsevol punt?
5. Tutor: mira d’entendre l’enunciat del problema [Cognitive message of level one]
6. Aleix: Ah...sí [he constructs a point P on the median and the parallel lines through P to the sides AC and AB, and he constructs the points E and F with the intersection tool]
Aleix observes the static figure on the screen and conjectures, without using measure tools, that the segments EM and MF have the same length. He does not validate the conjecture with the dragging tool, because he starts to verify the conjecture deductively. He has identified in the figure the inner similar triangle as we can see in the following paragraph.

7. La relació entre EM i MF és que són iguals [Figure 4.39]
8. Tutor: Com ho podries justificar?
9. Aleix: Perquè els triangles EPF i BAC són proporcionals….

[...] 
12. Aleix: vale, té tres costats paral·lels per tant són proporcionals. En un triangle sí que val [analogy with the case of rectangles in the root problem: he tried to apply this criteria to quadrilaterals] 

Aleix justifies deductively that PM is the median of the triangle EPF. He is aware of the fact that the reasoning is valid for any point P of the median and any triangle ABC. We code this fragment with a (+) score concerning the deductive competence and the visualization competence.

For the resolution of the second question, Aleix applies the heuristic strategy of considering the problem solved. The following fragment shows again that Aleix understands the logical dependence of the elements of the GGB figure. Dragging the point P, the points E and F are indirectly dragging and the ratios between homolog segments are preserved. He drags the point P to obtain an approximate trisection of the base BC through perceptual approach, and he then tries to define the geometric properties of the point P to construct a robust figure.

17. Aleix: vale, a veure, com són proporcionals, he de fer tal proporció que la base de EPF sigui una tercera part de la base de BC….
18. Aleix: Ja que són triangles proporcionals, podem disminuir un costat. Per exemple l’alçada... [He points at AM]
19. Tutor: És l’alçada? [Cognitive message of level one]
20. Aleix: No, no. L’angle no és de 90 graus. És la mitjana
Aleix does not use dragging for adjusting to obtain the position of the point P on the median. He starts an execution phase by trying to construct a scaled copy of the triangle ABC with a factor scale one third. He identifies that scaling the median (varying the position of the free point P) the other dependent objects of the triangle EMF will be scaled (lines 17-18). As we will see, in this execution phase, Aleix overcomes a visual-algebraic difficulty and the orchestrated use of GeoGebra has a key role.

Aleix tries to construct a robust figure and searches in the toolbar, examining all the tools (Figure 4.40). Aleix’s behavior shows that he has the intuition of considering a geometric transformation, dilatation, as we can see in the figure below (Figure 4.40). He considers applying the tool dilatation.

![Figure 4.40: Exhaustive exploration of the tools. He selects the tool dilatation on two occasions](image)

As we can see in the paragraph below, he tries to find another tool rather than dilatation. He focuses on constructing a point P that trisects the median.

21. Aleix: per tal que un costat sigui la tercera part de BC jo he de fer que AP sigui una tercera part de...
   [Aleix searches in the toolbar the different tools and observes all the tools in each icon. He returns to the tool dilatation twice but he abandons it] (Figure 4.40)
24. Aleix: només es pot fer amb les eines que es veuen aquí?
25. Tutor: Com ho faries?
26. Aleix: No sé…
27. [Aleix remains in silence observing again the tools]
28. Aleix: es podria fer amb circumferencia…
29. Tutor: Com ho faries?
30. [Aleix clicks on the tool circumference given the centre and a point and then on the vertex A as the centre, and he drags the second point along the median to define visually the circumference]

![Figure 4.41: Tool circle given the centre (A) and a point (point on the median)](image)
Aleix first chooses the tool circle, given the centre and a point, but he discards this option as he should trisect the segment through perceptual approach (Figure 4.41). He stares again in silence at the tools. Aleix tries to apply the tool circle given the centre and the radius, but he is not sure about how to define the radius, as the length of the median is a parameter.

31. Tutor: tracta de considerar altres eines
32. Aleix: pues el mateix però… [Circumference given the centre and the radius]
33. Aleix: aquí es pot posar un terç de a? [He refers to the segment AM]
34. Tutor: sí
35. [Aleix inserts in the input box the expression 1/3 a for the radius and obtains a circumference of centre A and radius 1/3 a. [ He observes the figure obtained]
36. Aleix: Vale, això no sembla ser el resultat! [He erases the previous circumference]
37. Tutor: repassa les construccions fetes

Through perceptual approach, Aleix refutes the construction. Firstly, he observes that the length of the segment outside the circumference is less than one third of the median (Figure 4.42). Secondly, he drags the point P, to the intersection point, and observes through perceptual approach that the segment EF is not a third of the base (figure 4.43). Aleix starts a local evaluation phase.

During the local evaluation phase, Aleix carries out the following validation techniques:
- He checks the validity of the expression 1/3 by constructing the circumference of centre P and radius 1/3
- He checks the equation of the circumference in the algebraic window, identifying variations on the centre and the radius in the algebraic expression, while he drags the point P along the median
- Finally he uses the drag tool to validate the construction. He obtains a counterexample by dragging the vertex A of the triangle in a wide range. As he notices that the radius of the circumference is invariant while dragging A, he deduces that the parameter ‘a’ does not represent the length of the median (segment d).
Finally, he constructs the circumference of centre A and radius d/3 (one third of the length of the median AM). He refutes again the result, as we can see in the following paragraph.

46. Aleix: tampoc sembla ser el resultat! És massa petit…
47. [Aleix drags the vertices of the triangle and refutes the previous construction]
48. Aleix: Ah! Vale! Primer...per a que la circumferència... per a que l’espai que no avarca sigui un terç, he de posar dos terços!
49. Aleix inserts the radius 2/3d and defines the intersection point I of the median and the circumference. He drags the point P until P=I and defines the segments BE, EF, FC]

Aleix visualizes the algebraic-geometric relationships of segments with the mediation of the software. He applies a new validation technique, using the tool ‘relation between objects’ to compare the length of the segments BE, EF and FC obtained. He uses this validation technique because he is certain about the construction. Nevertheless, the software does not assign the equality relation to the lengths of the segments EF and FC. As Aleix is convinced of the validity of his deductive reasoning, he deduces that these slight differences are due to round-off errors. To be sure about this fact, he drags the vertices of the triangle A, B, C in a wide area, considering also degenerate cases, to observe if the round-off error can be ignored (Figure 4.44). He finally concludes: “Aleix: aquests dos són iguals (BE and FC) però EF és una mica més petit. Suposo que és degut a com he construït la circumferència...”.

![Diagram](image-url)
b) Aleix’s resolution process of the median problem

For the resolution of this problem, Aleix tries three different strategies: the trigonometric strategy based on working out angles, the strategy based on justifying the equalities of heights as an equivalent problem, and the strategy based on comparing areas (equicomplementary dissection rules). The role of GGB and the tutor’s orchestration are relevant for the shift from one strategy to another, as we will see in the next paragraphs. After reading the statement of the problem, Aleix draws a figure with paper-and-pencil (Figure 4.46). There are relevant aspects in this figure. On the one hand, Aleix does not label the points, he just marks a median, a point on the median and two inner triangles. This shows that he has a deep understanding of the statement of the problem (symmetry of the statement) and a high capacity of abstraction for a student of this age. The marks made on the segments reveal a change of anchorage from discursive to visual; he associates the statement of the problem (definition of median) with the configuration (Torregrosa & Quesada, 2008). The student will construct a triangle with GGB with the same orientation than the paper-and-pencil one. As GGB label the vertices, other students have considered the oblique median AM (Figure 4.47). These students have introduced the triangle with the polygon tool directly. Instead Aleix will introduce the vertices’ coordinates in the input field with their labels. He considers integer coordinates and constructs two points on the x-axis, but these points are not linked to the x axis, as in the case of Marta and Guillem.
On the other hand, the triangle represented by Aleix is close to an isosceles triangle. Aleix constructs with GGB the triangle of vertices $A = (0, 4)$, $B = (-3, 0)$ and $C = (4, 0)$. This first representation will suggest to Aleix, through perceptual approach, that the two inner triangles (APC and PBC) are congruent. He remarks that the triangles have a common side (mark on the figure 4.46) and tries to justify that they have two congruent angles introducing parallel lines (mental operative apprehension). We conjecture that he tries to apply the criterion ASA, but he does not mention it. The tutor’s cognitive message (line 10 of the protocol) makes Aleix consider other triangles, but he just drags the vertex $A$ locally (Figure 4.48 and figure 4.49). Nevertheless, he refutes the conjecture $C_1$ considering the triangle obtained as a counterexample. The tutor’s message fosters the development of the instrumented scheme ‘dragging to find a counterexample’.

Aleix abandons the previous strategy and starts an exploration phase with GGB. The geometric and the algebraic window combined with continuous dragging (Olivero, 2002) provide Aleix with perceptual evidence that the area and the length of the base are equal for both triangles. He deduces that if the areas are equal the heights have to be equal. The immediate perceptual approach leads Aleix to consider wrongly that $BM$ is the height of the triangles $ABM$ and $AMC$ (Figure 4.46), and he justifies the equality of areas:

\[
\begin{align*}
MB &= MC \\
\text{commonBase} &
\rightarrow \text{AreaFormula} \\
BM, MC &
\rightarrow \text{heights} \\
(BPA) &= (APC)
\end{align*}
\]
The tutor gives Aleix a validation message. Aleix reacts by revising his previous deduction and by observing the static figure on the screen. He notices that BM and the AM are not perpendicular, thus BM is not a height of the triangle. This may be due to a difficulty in visualizing the exterior heights of the triangle.

![Figure 4.50: Construction of the heights (Aleix)](image)

Aleix goes on with the same solving strategy to construct the heights of the triangles, but he does not construct the feet of the heights (Figure 4.50). This fact may obstruct the visualization of the inner right-angled triangles formed by the heights, the median and the triangle’s side BC (Figure 4.50). He tries again a trigonometric strategy, reasoning on the static figure to justify the equality of areas. Aleix is a high-achieving student and is confident about his ability to manipulate trigonometric formula. He may try to consider this longer way as a last resort. Nevertheless, the data of the problem make it difficult to connect the angles and the sides of the triangles APB and APC. This time the tutor’s message will influence Aleix’s solving strategy and will foster the emergence of instrumented schemes. He states: ‘Tracta de buscar invariants amb l’ajuda de GGB’ (line 22 of the protocol). Aleix is reticent to abandon the previous strategy, but finally, after another tutor’s message (‘Intenta descompondre el triangle en altres triangles’, line 24 of the protocol), he starts a new exploration phase.

He explores via continuous dragging (Olivero, 2002). There is a relevant aspect which proves the development of instrumented schemes. Aleix drags all the ‘draggable’ elements of the figure over a wide area, looking for geometric invariants and algebraic invariants. For instance, he drags the vertex A, continuously changing the orientation of the triangle (Figure 4.51). In this case, the use of GGB helps Aleix to extract two inner triangles and conjecture that the triangles have the same area. This leads to the
construction of a deductive proof based on comparing areas (equicomplementary dissection rules, Figure 4.52).

**Figure 4.51:** *cinema dragging* of the vertex A (the point D corresponds to the point M of the problem's statement)

**Figure 4.52:** Aleix’s worksheet (median problem)

**c) Aleix’s resolution process of the quadrilateral problem**

For the resolution of the quadrilateral problem, Aleix follows part of two resolution approaches. Through reconfigurative operative apprehension, he considers a degenerate case (only one triangle), but he does not follow the strategy based on considering particular cases. He uses reconfigurative operative apprehension to state a conjecture and then he proves the equality of areas by using the strategy based on obtaining equality of ratios to compare the base and the height of the triangles.
In the familiarization phase, Aleix draws a triangle (Figure 4.53) and draws marks on the equal segments (change of anchorage from discursive to visual). The position of the triangle on the worksheet obstructs the visualization of the areas to compare, and Aleix starts an exploration phase changing the orientation of the worksheet (Figure 4.54). Before stating a conjecture, he tries to analyse the figure obtained.

Figure 4.53: Aleix’s triangle

Figure 4.54: Change of orientation of the worksheet and second triangle

During the analysis/exploration phase, Aleix draws a second triangle (Figure 4.54) and considers a different position of the point P. He identifies that the line MN is parallel to the base AC of the triangle and draws the heights of the triangles APM and PNC through the vertices M and N. He draws marks on the heights to show equality of lengths (discursive apprehension). During this analysis/exploration phase, Aleix conjectures the equality of areas. He states ‘Llavors ...he de veure que les àrees són iguals (line 17)’. As Aleix does not express how he has conjectured the equality of areas, the tutor asks him to give more details, as we can see in the paragraph below.

34. Tutor: Com has arribat a la conjectura de que les àrees són iguals?
35. Aleix: He vist que l’àrea d’això [PMN] no depèn de P. Llavors aquesta àrea és constant i com aquesta és constant, aquesta també [(CPN)+ (PMA)]. La part de sota és independent de la part de dalt (PAM).
36. Aleix: També he considerat un cas particular: P el punt mig. Com que no importa on estigui P puc considerer aquest cas. He vist que totes les altures són iguals per paral·leles llavors he suposat que les àrees són iguals [...]
We observe that Aleix is close to the strategy based on particularization. He states that the areas are constant and conjectures for a particular case (particular equilateral triangle and particular point P - midpoint) that the areas are equal. Nevertheless, he does not generalize the property because he has proved that the area is constant when P varies along the base in the case of an equilateral triangle (Figure 4.55). He is aware of the fact that he can not generalize the result obtained to any triangle.

![Figure 4.55: Particular case- P midpoint and equilateral triangle. He does not consider labels for the points](image)

Aleix has difficulties in proving Thales reciprocal, as we can see in the paragraph below. He uses it as an accepted theorem. The tutor gives Aleix a cognitive message to foster the emergence of strategies of proving, such as proving by contradiction.

8. Aleix: […] Llavors MN és par·llela a AC
9. Tutor: Com ho justificaries
10. Aleix: en un triangle els punts mitjós estan en la paral·llela a la base AC
11. Tutor: Ho sabries justificar?
12. Aleix: L’alçada és la mateixa, com que sé que l’alçada és perpendicular i la longitud és la mateixa, faig la recta per dos punts i és paral·llela…o sinó amb Thales
13. Aleix: si les rectes són paral·leles i …
15. Tutor: Que passaria si les rectes no fossin paral·leles? [Cognitive message of level two]
16. Aleix: No serien segments iguals, suposo…

Aleix considers Thales reciprocal as an accepted theorem and tries to justify deductively the equality of areas. He states “Llavors, he de veure que les àrees són iguals” (line 17). He starts an execution phase with the implicit plan of applying the area formula of the triangle to compare both areas. He visualizes through reconfigurative apprehension that both triangles (AMP and PNC) can be transformed in an equivalent shape (triangle of base AC). Nevertheless, he does not visualize that the area of this equivalent shape is half the area of the outside triangle ABC. This is due to the fact that he focuses on comparing the areas of the inner shapes and he does not pay attention to the exterior triangle. He also deduces that the quadrilateral can be split into two equivalent triangles. Using these two inferences, he expresses both areas as \( \frac{4C}{2} h \) and \( \frac{MN}{2} \times h \) (Figure 4.56).

Aleix constructs a proof by equivalence. He starts with the goal proposition and proceeds in a finite number of steps by equivalence until he gets a true proposition. He then proceeds backwards to obtain the proof of the desired result (equality of areas), as we can see in the paragraph below. We code this paragraph with a (+++) score in the deductive competence.
25. Aleix: Podem agafar aquests triangles [AMP i PNC] i dir que és un sol triangle imaginari de base AC i multiplies la base AC per l'alçada i divideixes per dos.
26. Tutor: que vols dir amb triangle imaginari?
27. Aleix: bé, tinc dos espais [both triangles] i dic que només ho considero com un sol espai
28. Aleix: i aquí [APM and PNM] també tenen la mateixa base MN i la mateixa alçada
29. Aleix: Suposem que les àrees són iguals
30. Aleix: Per Thales tenim que MN és un mig de AC
31. Aleix: ara igualo les àrees i vaig simplificant a veure que queda
32. Aleix: [simplifying he obtains MN = $\frac{1}{2} AC$ ].
33. Aleix: Com això per Thales és correcte, i simplificant les fòrmules de les àrees surt el mateix llavors també és correcte

Figure 4.56: proof by equivalence of the equality of areas (Aleix)

Finally, the tutor asks Aleix to explain with paper-and-pencil the resolution strategy for the quadrilateral problem. He elaborates a deductive justification using implicitly that the median splits the triangle into two triangles of equal area (Figure 4.57).

Figure 4.57: Aleix’s worksheet (quadrilateral problem)
4.2.2 Feed-forward concerning the local HLT

In this section we report the feed-forward concerning the three students’ resolution processes of the scaled triangles problem, the median problem and the quadrilateral problem. We consider firstly Marta’s resolution processes, then Guillem’s resolution processes and finally Aleix’s resolution process.

4.2.2.1 Feed-forward concerning Marta’s local HLT
We report the feed-forward concerning Marta’s resolution process of the scaled triangles problem, the median problem and the quadrilateral problem.

a) Feed-forward concerning the scaled triangles problem
Marta has difficulties in discerning properties of the figure and properties that are only spatial properties of the drawing that can not be considered as hypothesis. For example, the following paragraph shows this difficulty. We code this paragraph with a (o) score concerning the visualization competence, as she has difficulties in discerning properties of the figure and the drawing, but she partially overcomes these difficulties (line 10).

5. Marta: E i F són punts mitjors
6. Tutor: No ho diu l’enunciat [Cognitive message of level 2]
7. Marta: Aquí sí que està al mig...Però no té perquè ser així.
8. Marta: Però és que aquí sí que està al mig...Es veu en el dibuix que E i F són punts mitjors.
9. Tutor: no ho saps [cognitive message of level 2]
10. Marta: Ah, no! Mira. On he de situar P per obtenir BE = EF = FC. Llavors no pot estar al punt mig....

She does not react to the tutor’s cognitive message and still considers that E is the midpoint (line 8). She does not pay attention to the grid to appreciate visually that E is not the midpoint. Moreover, it is easy to appreciate visually in the statement’s figure that P is not the midpoint of the median. Considering Thales property, if E was the midpoint of the segment BM, P should be the midpoint of the median AM. This difficulty in interpreting the statement’s figure may be due to the fact that Marta does not discern the inner triangle ABM and has difficulties in understanding the dependency between the points E, F and P in the familiarization phase. Finally, she reacts to the tutor’s message and pays attention to the statement of the problem (line 10). Mercè understands the existential quantifier (existence of the point P that trisects the side BC), but misunderstands the concept of quantifier: “P any point in the median” in the first question. She does not consider that the equality EM=MF stands for any point P of the median, thus she tries first to construct the points E and F that verify the statement of the second question, BE=EF=FB, and then construct the point P, as we can see in the following paragraph.

10. Marta: Ah, no! Mira. On he de situar P per obtenir BE = EF = FC. Llavors no pot estar al punt mig....
11. Marta: Aquestes paral·leles han d’estar més avall. He de fer la mateixa figura pels dos apartats [She considers the same position of the point P for both questions: unique point P that verifies EM=MF and trisection of the segment BC]
12. Marta: Divideixo en tres parts... [The base BC of the triangle] i ja sé on està P.

We assign a (o) code to the above paragraph concerning the structural competence, because she does understand adequately the logical structure of the problem. Despite
understanding the existential quantifier (point P that trisects), she does not understand that the property EM=MF stands for any point P of the median. Marta visualizes that the point P can not be the midpoint of the median to obtain the trisection of the side BC (line 11), but she does not visualize that the point P is a point of trisection of the median. We assign to the above paragraph the (o) score concerning the visualization competence.

She is not able to construct with GGB a robust diagram and then she uses “dragging for adjusting” to obtain the point E. This is a technical obstacle, as she is aware of the fact that the measure of the segment BE should be 2 units if the measure of the segment BC is 6 units, and she visualizes the position of the point P on the median: “Aquestes paral·leles han d’estar més avall...”. Nevertheless, we conjecture that she may have difficulties in trisecting a segment with ruler and compass, because the transfer of this paper-and-pencil technique to the GGB technique is transparent. She does not consider the technique of introducing the coordinates of the points E and F using algebraic properties. As she has considered A(0,0) and B(b,0), she may introduce E(\(\frac{b}{3},0\)) and F= (\(2\frac{b}{3},0\)). We conjecture that she does not bear in mind the introduction of points in the input field, as she does not use it in any problem.

Despite the fact that she is aware of Thales property, she only uses it as a one-step procedure to obtain the length of a segment. For example, she has difficulties in dividing a segment applying Thales property or in deducing an equality of segments applying twice Thales property (lines 31-38). Moreover, she tries to apply blindly Thales property without having a solving strategy in mind. She states “triangles és Thales o Pythagoras”. This belief is an obstacle for the visualization process, as she focuses only on triangles in the usual Thales configuration. As Duval (1994, pp. 121-122) states: “\textit{Mais, en réalité, pour beaucoup d’élèves les figures ne fonctionnent pas du tout comme cet outil heuristique lors des phases de recherche. La simple vue d’une figure semble exclure le regard mathématique sur cette figure. Deux types de difficultés persistantes sont, en effet, couramment constatés, aux différents niveaux de la scolarité:}

- \textit{la résistance à se détacher des formes et des propriétés visuellement reconnues du premier coup d’œil: la figure constitue alors une donnée intuitive qui se suffit à elle même et qui rend inutile ou absurde, toute exigence de démonstration.}

- \textit{L’incapacité à voir dans une figure, c'est-à-dire à discerner des éléments de solutions possibles à un problème posé: cela supposerait que l’attention ce focalise sur certaines parties de la figure plutôt que sur d’autres, ou que la figure soit éventuellement Guillemhie de tracés supplémentaires. Or il y a tellement d’entrées possibles dans une figure que le choix de l’une d’elle paraît arbitraire, et surtout ce choix semble présupposer que l’on connaisse déjà la solution cherchée! ’’}

Marta senses that P is the centroid of the triangle. She may relate the medians of a triangle with the centroid. Nevertheless, she is not able to visualize that G is a point of trisection of the median AM. She tries to obtain properties of the centroid and takes measurements of the segments AG and GM to find quantitative relations between these segments. But this \textit{wandering measuring} (Olivero, 2002) does not help Marta, as she trusts that measurements with GGB are exact and does not conjecture that AG=2GM.
This is a technical difficulty in interpreting the round-off error. The tutor helps Marta to interpret this fact.

The most relevant obstacle in this problem is a visual-algebraic obstacle. Marta is not able to relate the trisection of the segment BM with the trisection of the base BC, which is twice the segment BC. She is not aware of the fact that if she trisects the segment, 3EM =BM, then 2EM=EF is a trisection of the segment BC. The purpose of the first question is to help the students to observe this fact, but Marta does not relate both questions. Another proof of the same obstacle is the fact that Marta has obtained the relation AG=2GM, but she does not see the relation AM=3GM. The tutor helps Marta to visualize this property, but she is not able to relate the trisection of the median AM with the trisection of the segment BM and thus BC. Marta is not able to construct a proof.

In fact, she also has difficulties in proving the equality EM=MF as she applies Thales property blindly. She tries to justify the equality of the expressions obtained for EM and MF by considering the measures in the particular figure (naïve justification). We condense in the following table (Table 4.5) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the scaled triangles problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Visualization</strong></td>
<td>(o) Distinction between drawing and figure (lines 5-10 and lines 42-45)</td>
</tr>
<tr>
<td></td>
<td>(o) Correct visualization of the position of P in the median (she identifies the centroid) but not the ratio 1:3 (lines 10-12)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing the relation between the trisection of the segment BM with the trisection of the base BC, which is twice the segment BM (implicit in the resolution strategy and in the worksheet’s figure)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing the relation between the ratio 1:2 and the ratio 1:3 in the segment AM (lines 54-58)</td>
</tr>
<tr>
<td><strong>Structural</strong></td>
<td>(o) Distinction between drawing and figure (lines 5-10 and lines 42-45)</td>
</tr>
<tr>
<td></td>
<td>(o) She identifies intuitively that the point P is the centroid, but she is not able to identify its properties (lines 50-57)</td>
</tr>
<tr>
<td></td>
<td>(o) She applies two-steps Thales with the help of the tutor (lines 39-41)</td>
</tr>
<tr>
<td></td>
<td>(+) She understands the logical structure of the problem (two quantifiers)</td>
</tr>
<tr>
<td><strong>Instrumental</strong></td>
<td>(o) She understands the necessity of proof despite proposing a naïve justification (lines 36-38)</td>
</tr>
<tr>
<td></td>
<td>(+) She tries to state a conjecture about the position of the point P (lines 46-51)</td>
</tr>
<tr>
<td><strong>Deduction</strong></td>
<td>(+) Deductive justification for the first question</td>
</tr>
<tr>
<td></td>
<td>(-) Naïve justification for the second question. She stays at a spatial-graphical field (written protocols)</td>
</tr>
</tbody>
</table>

*Table 4.5: Competences coding system (Marta)*

We report now the acquisition degree of instrumentation. We have identified the following difficulties with instrumented schemes:

- Instrumented scheme that consists in dragging combined with 2-steps visual ratio comparison. This scheme requires: a) the awareness of Thales property, b) the mental step of extracting two inside triangles and comparing the common side, c) the awareness of considering congruent sides. Those mental activities give meaning to technical actions, such as: i) selecting the point P to be dragged along the median, ii) observing visually or defining the measure of the segments on the geometric/algebraic window.

Marta has difficulties in carrying out this scheme, as she has difficulties with the steps b) and c). In fact, she applies the simple scheme “dragging combined with visual
ratios comparison in one step”. She obtains an equality of ratios comparing congruent sides of two triangles in usual Thales configuration (Figure 4.58). The tutor’s orchestration helps Marta to consider the composed scheme. This obstacle is related to the paper-and-pencil obstacle: she is not able to extract both triangles and find an algebraic relation considering its common side.

![Figure 4.58: Simple instrumented scheme: ratio comparison (Marta)](image)

- **Simple scheme: interpreting round-off error to validate a property of a figure**

It involves the technical skill of constructing robust diagrams (managing construction tools and techniques). It implies the mental image of the difference between Euclidean plane and GGB plane. As stated by Olivero and Robutti (2007), geometric figures in DGS are constructed according to Euclidean properties, but the DGS plane does not coincide with the Euclidean plane: the DGS plane has a finite number of elements, i.e. pixels, which are the smallest units in the computer. Measurements are visualized with a certain degree which a certain degree of approximation that can be set by the user. The number of decimal digits to be visualized on the screen can be chosen by the user. This double nature of measuring resembles the paper-and-pencil environment, because one can draw and measure with ruler and compass, but in general students are aware of the precision of a ruler. Marta has difficulties with this scheme as she trusts that measurements with GGB are exact.

Marta has also technical difficulties with: a) applying the tool parallel line to two segments, b) reporting distances to construct a robust figure (she does not consider the tool circle given the center and the radius), c) trisection of a segment, and d) round-off error obstacles. She does not use the dragging option to validate the conjectures in other triangles (scheme dragging for validating a conjecture), but it may be due to the fact that the triangle of the statement is represented on a grid. She uses the drag tool to: a) report distances, and b) test a conjecture, but she does not use the drag tool to modify the constructed figure. Moreover, she is not aware of the fact that the point A(0,0) is non-draggable and B(0,6) is linked to the x-axis.

**b) Feed-forward concerning the median problem**

Marta has difficulties in visualizing the exterior heights of the triangle and also in following the strategy based on comparing heights and bases to compare the areas. The use of GGB combined with the tutor’s orchestration helps Marta to understand the strategy based on justifying that the heights of the triangles ACP and APB (corresponding to the common side AP) have the same length. She understands the need
of proving and tries to justify this equality through congruence of triangles. Nevertheless, she has a misconception with the congruence criteria ASA. She considers that two equal angles and an equal side is a sufficient condition to assure the congruence of triangles. We condense in the following table (Table 4.6) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the median problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(-) Difficulty in visualizing exterior heights of a triangle (lines 8-16)</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure (counter-example, lines 27-30)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (she extracts two triangles from the initial configuration) (lines 22-25)</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure (lines 27-30)</td>
</tr>
<tr>
<td>Structural</td>
<td>(+) Understanding of quantifier: ‘P any point of the median’, ‘any triangle’</td>
</tr>
<tr>
<td></td>
<td>(+) (Discursive apprehension) Congruence of Alternate angles (lines 20-22)</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure (counterexample, lines 27-30)</td>
</tr>
<tr>
<td></td>
<td>(o) Congruence criteria ASA (she does not consider the adjacent side S but she identifies in the figure that both hypotenuses are congruent) (lines 22-25)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(o) Construction of an isosceles triangle as general case (but proof not dependent on the drawing) (lines 22-30)</td>
</tr>
<tr>
<td></td>
<td>(+) She understands the necessity of proof (lines 4-7)</td>
</tr>
<tr>
<td></td>
<td>(+) Search for a counterexample (lines 27-30)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) Attempts a sequence of deductive steps (correct reasoning. She uses the criteria SAS as tool of her deductive reasoning but the criteria is applied wrongly).</td>
</tr>
</tbody>
</table>

Table 4.6: Competences’ coding system (Marta)

We report now the acquisition degree of instrumentation. Marta tries to construct robust figures and has no difficulties with elementary tools for constructions. In fact, the technical difficulty with the intersection tool is related to the difficulty in visualizing the exterior height of a triangle. We observe the development of the following instrumented schemes:

- dragging combined with measure tools to validate a conjecture: equality of areas
  \[(APB) = (APC)\]
  She is aware of the fact that the property has to be verified for all the points P of the median. Nevertheless, she does not consider dragging the vertices of the triangle ABC.

- dragging combined with measure tools to validate a conjecture: equidistance of the points B and C to the median AM (equality of height’s measures)
  She is aware of the fact that the property has to be verified for all the points P of the median. Nevertheless, she does not consider dragging the vertices of the triangle ABC.

- dragging to obtain a counterexample for the statement “the median split the triangle in two congruent triangles”. This scheme requires the mental awareness of the distinction between drawing and figure.
  For the first time she considers dragging the vertices of the initial triangle ABC. She does not notice that the point A is un-draggable and the point B is linked to the x-axis. She uses photo-dragging (Olivero, 2002) but in this case the purpose of the scheme is finding a concrete position of the elements of the figure.
c) Feed-forward concerning the quadrilateral problem

Marta uses particular cases (right-angled triangles) to state conjectures and to visualize geometric properties, but she is aware of the fact that she has to generalize to other triangles the geometric properties observed (line 43 of the protocol). She understands the concept of quantifier ‘P any point of the side AC’, but she considers the initial triangle, which is close to an isosceles triangle, as a general case. She tries to find geometric properties on the figure, but she obtains properties that are related to the consideration of an isosceles triangle (line 8 of the protocol). She still has visual-algebraic difficulties (visualization of the resolution of the linear equation). Nevertheless, she is aware of the need of proving. Marta has difficulties in justifying the equality of heights as she does not consider the concept of height as distance from a point to a line. In fact, she is not sure about the equality of heights and considers through perceptual approach that the heights may be different. Nevertheless, she is aware of the lack of precision of the drawing. She considers the distinction between drawing and figure.

She also has difficulties in proving the midpoints theorem, but finally with the help of the tutor she uses Thales theorem to justify the midpoints theorem.

For the first time she mentions the relation between Thales configuration and similar triangles. We conjecture that in the previous problems she was not able to identify non usual Thales configuration (problem of the median) or triangles in a homothetic relation (scaled triangles problem). Finally, we observe that she does not mention that if the areas are equal, they are half of the outside triangle’s area. This may be again an algebraic-visual obstacle. We condense in the following table (Table 4.7) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the quadrilateral problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Visualization</strong></td>
<td>(+) Visualization of variations of the point P (Marta visualizes particular cases)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension: she discerns two equivalent triangles inside the quadrilateral</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing that the addition of areas is constant (algebraic-visual distinction)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing Thales configuration in a non-standard situation (an auxiliary parallel line is required)</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure (she discerns geometric properties)</td>
</tr>
<tr>
<td><strong>Structural</strong></td>
<td>(+) Distinction between drawing and figure (lines 28-31)</td>
</tr>
<tr>
<td></td>
<td>(+) Understanding of the quantifiers ‘any point P’ (Figures 4.17, 4.19, 4.20 and lines 3,4), ‘any triangle’</td>
</tr>
<tr>
<td></td>
<td>(+) Relation between parallelism and proportional segments (18-25)</td>
</tr>
<tr>
<td></td>
<td>(o) Relation between Thales theorem and similar triangles (usual configuration) (lines 50-52, Figure 4.19)</td>
</tr>
<tr>
<td><strong>Instrumental</strong></td>
<td>(o) Counterexample (based on perceptual properties of the drawing) (lines 28-30, Figure 17)</td>
</tr>
<tr>
<td></td>
<td>(+) Conjecture (equality of areas) with the help of the tutor (lines 47-48)</td>
</tr>
<tr>
<td></td>
<td>(o) Conjecture (which is only verified for a particular triangle) (Figure 4.16, line 5)</td>
</tr>
<tr>
<td><strong>Deduction</strong></td>
<td>(+) Deductive reasoning (Figure 4.26, worksheet)</td>
</tr>
</tbody>
</table>

Table 4.7: Competences coding system (Marta)

4.2.2.2 Feed-forward concerning Guillem’s local HLT

We report the feed-forward concerning Guillem’s resolution process of the scaled triangles problem, the median problem and the quadrilateral problem.

a) Feed-forward concerning the scaled triangles problem
The figure, inserted in a grid, of the problem’s statement obstructs Guillem understanding of the problem. In the familiarization phase, he considers that he has to reproduce the figure with GeoGebra. Despite this difficulty, Guillem understands the statement of the problem and the concept of quantifier “P any point of the median”.

He has difficulties in applying Thales property as a two-step procedure and in visualizing a resolution strategy based on applying Thales theorem.

The most relevant obstacle in this problem is a visual-algebraic obstacle. He has difficulties in relating the trisection of the segment BM with the trisection of the segment BC, which is twice the segment BM. He is not aware of the fact that if he trisects the segment 3EM =BM, then 2EM=EF is a trisection of the segment BC. The purpose of the first question is to help the students to observe this fact, but Guillem does not relate both questions. He overcomes this obstacle by introducing another median as an auxiliary element. He uses the fact, justified empirically, that the centroid is a point of trisection of the medians. Then he applies Thales property and justifies the trisection with one-step procedure. We condense in the following table (Table 4.8) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the scaled triangles problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(-) Difficulty in visualizing the relation between the trisection of the segment BM with the trisection of the base BC which is twice the segment BM (resolution process)</td>
</tr>
<tr>
<td></td>
<td>(+) Difficulty in visualizing the ratio 1:3 in the segment AM (median) (28-35)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension: he introduces another median and parallel lines (28-31)</td>
</tr>
<tr>
<td>Structural</td>
<td>(+) Understanding of quantifier: ‘P any point of the median’ (lines 2-8)</td>
</tr>
<tr>
<td></td>
<td>(o) Thales property applied without a specific strategy (lines 20-22)</td>
</tr>
<tr>
<td></td>
<td>(+) He understands the necessity of proof (lines 15-19)</td>
</tr>
<tr>
<td></td>
<td>(+) He understands the necessity of proof (lines 20-22)</td>
</tr>
<tr>
<td></td>
<td>(+) He understands the necessity of proof (lines 28-31)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(+) He uses deductive reasoning by logical inferences (lines 25-27)</td>
</tr>
<tr>
<td></td>
<td>(+) Deduction (line 36 and worksheet). He bases his reasoning on the properties of the centroid, validated empirically</td>
</tr>
<tr>
<td>Deduction</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: Competences coding system (Guillem)

Guillem shows in this problem a low acquisition degree of instrumentation. He has difficulties with: a) applying the tool parallel line to two segments, b) reporting distances to construct a robust figure (he does not consider the tool circle given the center and the radius), c) interpreting round-off error, d) applying the tool hide and show elements (this obstacle leads Guillem to avoid the construction of auxiliary elements, line 60). He does not use the dragging option to validate conjectures in other triangles, but this may be due to the fact that the triangle of the statement is represented on a grid (line 1). He uses the drag tool: a) to report distances, b) to test a conjecture, but he does not use the drag tool to modify the constructed figure (he does not drag the vertices of the triangle). He uses the drag tool to validate that the geometric property is maintained. For example, he drags the point P along the median AM to check the equality of areas (linked dragging). We observe that he uses this tool in the familiarization and analysis phases. In the exploration phase, he reasons on the static figure. Guillem uses GGB proactively. He appears to have a plan in mind about how to use GGB. For example, he decides to define auxiliary elements having certain expectations of what he wants to observe. He avoids using
measure tools to validate a conjecture; he validates his conjectures through dragging and visual perception.

We have identified the following instrumented scheme:

- Dragging to validate a conjecture: equidistance of E and F from M (M midpoint of the segment EF)

He has conjectured that EM=MF and drags the point P along the median to validate the conjecture. This scheme requires the awareness of the need of constructing robust figures (scheme to validate a construction) and the theorem-in-act: if the property is conserved through dragging all the independent elements then the property is true. It also requires the technical actions such that: i) select the elements to be dragged.

Guillem does not consider dragging all the elements of the figure. For example, he does not consider triangles other than the initial isosceles triangle. Moreover, he is not aware of the fact that the point A(0,0) is fixed and the point B(6,0) is linked to the x-axis.

b) Feed-forward concerning the median problem

Guillem has difficulties with the definition of height and also in visualizing the exterior heights of the triangle. The use of GGB combined with the tutor’s orchestration help Guillem to define correctly and to visualize the exterior heights of the triangles. He understands the need of proving and follows the strategy based on justifying the equality of the heights’ length.

We condense in the following table (Table 4.9) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the median problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(+) Dynamic Visualization (varying P along the base) (lines 3-7)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing exterior heights of a triangle (lines 12-19)</td>
</tr>
<tr>
<td></td>
<td>(-) Immediate perceptual approach of the triangles as an obstacle for the geometric interpretation (wrong conjecture) (lines 12-13, Figure 4.35)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (he extracts two triangles from the initial configuration) (lines 23-25)</td>
</tr>
<tr>
<td>Structural</td>
<td>(-) Conception of the height as perpendicular line to the base of the triangle (instead of the line that contains the base).</td>
</tr>
<tr>
<td></td>
<td>(+) He understands the quantifier: ’P any point of the median’</td>
</tr>
<tr>
<td></td>
<td>(+) Search for conjectures and partial validation : Conjectures C₁ C₂ C₃ (lines 6.-10, 12-14)</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension [Trigonometry of the right-angled triangle to justify congruence of two inside triangles, properties of the area] (resolution process)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(o) Construction of an isosceles triangle as general case, he does not validate the geometric properties in other triangles (but proof not dependent on the drawing) (lines 1-4)</td>
</tr>
<tr>
<td></td>
<td>(+) He understands the necessity of proof (lines 8-10 and lines 23-24)</td>
</tr>
<tr>
<td></td>
<td>(+) Figural inference (line 9)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) Deductive reasoning to obtain a key lemma (lines 8-10)</td>
</tr>
<tr>
<td></td>
<td>(+) Deductive reasoning to elaborate a proof (lines 23-27)</td>
</tr>
</tbody>
</table>

Table 4.9: Competences coding system (Guillem)

Concerning the acquisition degree of instrumentation, Guillem tries to construct robust figures and has no difficulty with elementary tools for constructions. In fact, the technical
difficulty with the intersection tool is related to the difficulty in visualizing the exterior height of a triangle. He has a technical difficulty with the dragging tool (he clicks near the point P instead of clicking on the point P). He has difficulties in interpreting the motion, but the teacher’s orchestration helps him to understand this motion in terms of logical dependency within the geometrical context.

We observe the development of the following instrumented schemes:

- dragging combined with measure tools (area) to validate a conjecture: equality of areas (APB)=(APC)

He is aware of the fact that the property has to be verified for all the points P of the median. Nevertheless, he does not consider dragging the vertices of the triangle ABC. He uses \textit{cinema-dragging} (Olivero, 2002):

- dragging combined with perceptual approach tools to validate a conjecture—equidistance of the points B and C to the median AM (equality of height’s measures). Guillem avoids the use of measures and tries to validate the conjecture using \textit{cinema-dragging} (Olivero, 2002) combined with perceptual approach. In this particular case, the measure of the heights is not relevant as the heights are invariant when dragging the point P along the median AM. This is due to the property that the two vertices of a triangle are equidistant to the median from the third vertex.

- \textit{Cinema-dragging} (Olivero 2002) to analyse variations on the figure through motion. He uses this scheme to find invariant properties of the inner triangles ABP and ACP when dragging P along the median (equal areas, common base). This scheme requires the knowledge of the area formula for a triangle and the awareness of the correspondence base-height for the use of this formula. It also requires the visualization of the of the height correspondent to the common base. Guillem deduces that the heights have to be equal because the triangles have always the same area and a common variable base. Nevertheless he does not consider dragging the other vertices of the initial triangle ABC.

c) \textit{Feed-forward concerning the quadrilateral problem}
Guillem understands the logical structure of the quadrilateral problem (any triangle, any point P of the side BC), and for the first time he tries to draw a ‘generic triangle’. Moreover, he draws the triangle with an oblique orientation.
Guillem understands the concept of height as distance from a point to a line, and thus he considers the height as a perpendicular line to the base. We have already observed this fact in the median problem. Guillem tried to construct only one height, but he had technical obstacles (instrumentation of the scheme ‘dragging to observe invariants’) and more precisely with the distinction between free/dependent objects.
Guillem has difficulties in applying Thales theorem to compare transversal segments. This is due to the fact that he does not relate Thales theorem with similar triangles. He associates Thales theorem with the division of a segment.
Guillem has also difficulties in justifying the midpoint theorem (parallelism of the line through the midpoints). He states “\textit{Clar, aquí sense GeoGebra no ho saps,...a lo millor el punt mig està una mica més avall}”. We may interpret this sentence as a drawback of
using GGB; we may interpret that visually convincing evidence provided by GGB would replace the need of proving. When working with GGB, Guillem considers the instrumented technique to check parallelism, which consists in drawing a parallel line through the point M to the line AC, and the accurate drawing may convince him. Nevertheless, we have shown that Guillem understands the need of proving, but he needs conviction to start the proof of a conjecture. As stated by Mike de Villiers (1997), who analyses the synergy between using Dynamic Geometry Software and proving the conjectures that arise from its use, conviction is necessary for considering the search for a proof. Moreover the software may give insight into geometric behavior that can help with a proof. We deduce from Guillem’s statement that he is aware of the lack of precision of a paper-and-pencil drawing, but he does not consider the difference between the discrete GGB plane versus the continuous Euclidean plane. He is not aware of the fact that the number of points of the GGB plane is finite when he uses the GGB technique to validate parallelism (drawing a line through M to AC and checking visually that the point N belongs to the line). This technique would be valid if the software had symbolic calculus and if he checked the property on the algebraic window. We condense in the following table (Table 4.10) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Experimentation with the quadrilateral problem</th>
</tr>
</thead>
</table>
| Visualization | (+) He draws a “general” triangle in an oblique position (Figure 4.38)  
(+o) Operative apprehension (dynamic visualization) (5-7)  
o) Difficulty in visualizing Thales configuration (lines 47-51)  
(-) Difficulty in visualizing the relation between the areas of the inner figures and the outside triangle (resolution process, line 6, lines 41-45) |
| Structural | (+) Distinction between drawing and figure (lines 11-14)  
(+o) He understandands the logical structure of the problem (any triangle, any point) (Figure 4.36, lines 6-7, 11-14)  
(+o) Search for conjectures and effort to validate them (lines 3, 11-14)  
(+o) Discursive apprehension: Equivalence parallel lines and angle congruence, Thales theorem, properties of the area function.  
(+o) Conjecture (equal areas) |
| Instrumental | (+) Figural inference (lines 3-7)  
(+o) Search for key-lemmas (line 41-46)  
(+) Necessity of proof (lines 8-12)  
(+o) Figural inference (lines 41-46) |
| Deduction | (o) He is has difficulties in proving Thales reciprocal: case-based reasoning (lines 8-16)  
(+o) Deductive reasoning (41-49). |

Table 4.10: Competences coding system (Guillem)

4.2.2.3 Feed-forward concerning Aleix’s local HLT
We report the feed-forward concerning Aleix’s resolution process of the scaled triangles problem, the median problem and the quadrilateral problem.

a) Feed-forward concerning the scaled triangles problem
The resolution of this problem combined with the use of GeoGebra helps Aleix to visualize the algebraic relationships between quantities as segments. Moreover, he understands the correspondence between free/dependent GeoGebra object with the logical relations of the problem. Also, he is near to developing an awareness of the homothetic transformation H(M,1/3) between the triangles EPF and ABC. This notion
has already emerged in the root problem. Aleix identifies similar rectangles and observes that by multiplying the sides of the inner rectangle by the same factor he obtains the sides of the outside rectangle. In the root problem, the tutor’s intervention fosters an awareness of criteria for similarity of rectangles. Moreover, these rectangles are in a homothetic relation as the triangles of the scaled triangles problem. Aleix uses this knowledge in the scaled triangles problem. In the following table (Table 4.11) we present the scores assigned to each competence.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Exploration with the scaled triangles problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(+) operative apprehension (he extracts similar triangles)</td>
</tr>
<tr>
<td></td>
<td>(+) Partial visualization algebraic-geometric</td>
</tr>
<tr>
<td></td>
<td>(+) distinction between drawing and figure (lines 4-6)</td>
</tr>
<tr>
<td>Structural</td>
<td>(++) Deep understanding of the logical structure of the problem</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension: similarity criteria for triangles, similarity ratio</td>
</tr>
<tr>
<td></td>
<td>(+) distinction between drawing and figure (exploration of round-off error behavior by dragging to validate a geometric property)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(++) Spontaneous conjecturing and self-initiated effort to deductively verify the conjectures (line 7-12, 34-36, 46-49)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) Deductive justifications</td>
</tr>
</tbody>
</table>

Table 4.11: Competences coding system for the scaled triangles problem (Aleix)

The instrumentation degree for this problem is high. We have identified the development of several instrumented schemes.

- Dragging to validate a construction: Aleix drags the vertices of the triangle ABC to validate the construction of the point P which trisects the median. He also combines this scheme with the scheme dragging to verify equidistance through exploration of the round-off error.
- Dragging to analyse the algebraic-geometric variations.
- Dragging to obtain a counterexample

b) Feed-forward concerning the median problem

Aleix has difficulties in solving the median problem at the beginning of the resolution process and spends one hour working on this problem. He shifts from one strategy to another, but the principal obstacle is Aleix’s biased-conception that he has to solve this problem by using a trigonometric strategy. The use of GGB and the tutor’s orchestration have a key role in considering other resolution strategies.

The data of the problem are: the triangle, its median and a point on the median and the areas of the inner triangles formed with the median point. Aleix has in mind considering theorems relating sides and angles to tackle the problem, and struggles to find a trigonometric strategy based on working out angles and finding equations between angles and sides. Moreover, he considers a triangle that is close to an isosceles triangle, and the perceptual approach convinces him about the congruence of the triangles APC and APB that only states for an isosceles triangle. He conjectures that the inner triangles are congruent for any triangle ABC and does not try to validate the conjecture through dragging. Later, influenced by the tutor orchestration and the use of GGB, he refutes the conjecture (instrumented scheme ‘dragging to find a counterexample’).

The second strategy followed by Aleix consists of justifying that the exterior heights of the triangles have the same length. But he has difficulties in visualizing the exterior heights of the triangles and considers wrongly through perceptual approach that the base
BC is perpendicular to the median. This may be due to the fact that the initial triangle is close to an isosceles triangle in which the median is also the height. He constructs with GGB perpendicular lines but does not construct the feet of the exterior height. Moreover, the coordinate axes and the height of the triangle ABC from A overload the figure (Figure 4.59). These factors hinder the visualization of the inner congruent triangles formed by the perpendicular lines, the median and the side BC.

Figure 4.59: cinema dragging of the vertex A (Aleix)

The tutor suggests to Aleix that he explores with GGB and considers splitting the triangle ABC into other triangles (lines 22 and 24 of the protocol). The use of GGB has a key role in the resolution process. For the first time, during the exploration phase, he explores the motion dependency in terms of logical dependency within the geometrical context. He drags the vertex P along the median and the free vertices in a wide range of surface. He also tries to drag the polygons and to understand the motion dependency with the help of the algebraic window. There is a shift from the use of the static figure to reason to the emergence of instrumented schemes. He starts to use GGB as an exploration tool and a strategy searcher. In this case, dragging has a key role in forming the conjecture. As stated by Laborde, Kynigos, Hollebrands and Strässer. (2006), “Hölz (2001) distinguished two ways of using the mediation functions of the drag mode, a test mode on the one hand and a search mode on the other hand” (Laborde et al., 2006, p.287).

Moreover, the use of GGB suggests to Aleix other views of the concept of area. He connects the strategy based on the properties of area as a measure function, and the use of GGB helps him to arrive at the key idea of considering equivalent triangles. As stated by Torregrosa and Quesada (2008), the coordination between the visualization process and the discursive process enables the subject to obtain key ideas and use them to implement a theoretical discursive process, which generates a deductive proof as the solution of the geometry problem. In the case where the coordination factor enables the student to solve the problem, they distinguish two different types of process: Truncation and unproved...
conjecture. For truncation, “the coordination process provides the ‘idea’ of how to solve the problem deductively” (Torregrosa & Quesada, 2008, p. 326).

We condense in the following table (Table 4.12) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Exploration with the median problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visualization</td>
<td>(o) Distinction drawing-figure (line 3-7)</td>
</tr>
<tr>
<td></td>
<td>(-) Visualization of the exterior height of a triangle (lines 18-22)</td>
</tr>
<tr>
<td></td>
<td>(+) operative apprehension (he extracts two equivalent inner triangles)</td>
</tr>
<tr>
<td>Structural</td>
<td>(0) False conjecture through perceptual approach (congruence of triangles) based on a particular case. He does not validate the conjecture in other triangles (lines 3-7)</td>
</tr>
<tr>
<td></td>
<td>(+++) Understands the motion dependency in terms of logical dependency and the logical structure of the problem</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension: Criteria ASA, property of the median (it splits a triangle into two equivalent triangles), properties of the area function as measure function (resolution process)</td>
</tr>
<tr>
<td>Instrumental</td>
<td>(+) Necessity of proof (line 6)</td>
</tr>
<tr>
<td></td>
<td>(+) Counter example (dragging the elements of the triangle ABC to obtain different cases of triangles APC and APB)- refutation of the conjecture C1 (APC and APB are not congruent) (line 11)</td>
</tr>
<tr>
<td></td>
<td>(+) Key-lemma equivalent to the problem (line 23)</td>
</tr>
<tr>
<td></td>
<td>(+++) Search for geometric invariants through dragging in a large area all the elements (Figure 14-Cinema dragging)</td>
</tr>
<tr>
<td>Deduction</td>
<td>(+) Key-lemma (line 23)</td>
</tr>
<tr>
<td></td>
<td>(+++) Deduction rigorously formulated with paper and pencil</td>
</tr>
</tbody>
</table>

Table 4.12: Competences coding system for the median problem (Aleix)

Concerning the acquisition degree of instrumentation, Aleix has no have technical difficulties and bases his constructions on geometric properties. He avoids using measure tools from the toolbar (length, angles) and uses simultaneously the algebraic window and the geometric window.

We have identified the following instrumented schemes. One of these instrumented schemes was not considered in the a priori analysis.

- Simple scheme: interpreting round-off error to validate a property of the figure
  Aleix does not use measure tools from the toolbar, but he considers the information of the algebraic window. He is aware of the difference between the Euclidean plane and the GGB plane and is able to choose the number of decimal digits to be visualized on the screen.

- Dragging combined with perceptual approach to find a counterexample.
  It involves the technical skill of constructing robust diagrams and using the drag tool and the mental awareness of the distinction between dragging and figure.

- Dragging combined with perceptual approach to distinguish geometric properties of the figure (perpendicularity, congruence of triangles, equality of areas).
  He uses this scheme as a solving strategy searcher. This scheme requires the understanding the concept of invariant and thus the need of a variety of constructions rather than just one. It requires also the awareness of the fact that all dependent relationships are preserved through dragging.
For instance, in the resolution of the third problem, this scheme requires the mental step of extracting two inside equivalent triangles. Those mental activities give meaning to technical actions, such as: i) selecting the object (point, segment, polygon) to be dragged, ii) observing visually on the geometric/algebraic window looking for invariants.

- 'algebraic-geometric invariant guessing': cinema-dragging of an element while looking for algebraic invariants in the algebraic window

Aleix use this scheme to understand the motion dependency of the elements of the figure. As he tries to drag elements of higher dimension that points, the interpretation of the motion dependency is more complex. For instance, he clicks on a point inside the polygon AMC, labeled polygon3 (Figures 4.60 and 4.61) and he drags continuously the polygon first observing only the geometric window and then observing only the algebraic window looking for algebraic invariants. In this case, when dragging the polygon AMC, the length of the side AC remains invariant and the triangle ABC is enlarged, but the vertex B is fixed (Figures 4.60 and 4.61).

Also the areas are of both inner triangles are equal, and the ratio between the area of ABC and the addition of both areas (polygon 2 and polygon 3 in Figure 4.60) is preserved. This is due to the fact that the ratio defined by the point M in the median is preserved.

Figure 4.60: Algebraic invariants while dragging the polygon AMC: the point B, the segment b and the ratio of areas between the polygon 1 and the polygon 2 (Figure 4.61)
c) **Feed-forward concerning the quadrilateral problem**

A relevant aspect of the resolution process is that Aleix does not consider the area of the outside triangle to obtain the relation between the areas of the inner figures (quadrilateral and both triangles). Instead, he observes that the line MN splits the triangle into two constant areas (lines 34-36). Moreover, he considers a particularization strategy (degenerate case) but does not use the property that the median splits the triangle into two congruent triangles. This may be due to the fact that he particularizes on the triangle and on the point to state a conjecture.

The principal difficulty for Aleix is the elaboration of a proof by contradiction (Thales reciprocal), but we do not expect the awareness of this strategy of proving from students of this age. Moreover, he is aware of the strategy of proving by equivalence (lines 29-33), which provides evidence of a deep understanding of the deductive competence, and he starts to develop awareness of the proof by contradiction.

We condense in the following table (Table 4.13) scores that refer to the acquisition degrees of the geometrical competences.

<table>
<thead>
<tr>
<th>Competences</th>
<th>Exploration with the quadrilateral problem</th>
</tr>
</thead>
</table>
| **Visualization** | (+) Dynamic visualization: the area of MNP does not depend on P (lines 34-36)  
|                | (+++) Reconfigurative operative apprehension  (transformation of two shapes in an equivalent shape) (lines 25-28) |
| **Structural**  | (+++) Understanding of the logical structure of the problem (34-36)  
|                | (+) Discursive apprehension: Thales theorem, Thales reciprocal, definitions, properties of the area function as measure function |
| **Instrumental** | (+) Search for conjectures (lines 34-36)  
|                | (+) Search for conjecture-lemma |
| **Deduction**   | (+) Awareness of the distinction between implication and double equivalence, |
4.2.3 Transitions concerning the local HLT

In this section we reflect on the transitions in the acquisition degrees of geometric competences between consecutive problems for each student. We compare the assigned scores for each competence between different problems.

4.2.3.1 Transitions concerning Marta’s local HLT

In this section we reflect on the transitions concerning Marta’s learning trajectory.

a) Transitions between the root problem and the scaled triangles problem

In the tables (Tables 4.1 and 4.5), we show the scores referring to the acquisition degrees of the geometrical competences for the root problem and the scaled triangles problem. We compare the different acquisition degrees for each competence.

Visualization competence

For the visualization competence, there is a shift with respect to the root problem. Marta considers varying the position of the point P along the median, whereas in the root problem she understood it as a static problem. This shift is due to several reasons. Firstly, the statement of the second problem fosters the understanding of the dynamical structure: there is an existential quantifier (existence of the point P that trisects the segment). Secondly, the use of GeoGebra to construct the figure and the mediation of the tutor help Marta to use the drag tool and to visualize other configurations dragging the point P along the median.

Nevertheless, she still has visualization difficulties in the second problem. Whereas in the first problem she discerns congruent triangles, in the second problem she does not discern similar triangles. The most relevant obstacle in the second problem is a visual-algebraic obstacle. Marta does not relate the trisection of the median AM with the trisection of the segments BC. Also she has difficulties in visualizing the trisection of a segment, based on Thales theorem. She is misled by the orientation of the figure, as we can see in the written protocol. In this case, the use of GeoGebra does not help Marta to visualize the trisection of the triangles’ sides, because she avoids overcrowding the GGB figure. This may be due to a lack of confidence in the use of GeoGebra.

Finally, we there is a shift in the understanding of the distinction between the drawing and the figure. In the second problem, Marta understands in the context of the problem, and with the help of the tutor, the distinction of properties of the drawing and geometrical properties of the figure. The logical structure of the problem and the requirement of constructing the figure with GeoGebra facilitate this shift.

Structural competence

There are differences between both problems concerning the understanding of the logical structure of the problem. In the second problem, Marta understands it with the help of the tutor and the interaction of the task and the environment.

In the second problem Marta is able to justify the equality of segments by applying Thales theorem in a two-steps process, while in the first problem she was not able to
develop this resolution approach. This may be due to the fact that in the first problem she applied Thales theorem blindly (without a clear strategy in mind because she had not stated any conjecture). Also, in this problem she has to prove the equality of segments whereas in the first problem she had to prove the equality of areas. Nevertheless, for the second question, she conjectures that the point P has to be the centroid of the triangle but she is not able to identify the properties of the centroid on her own. This is related with the visual-algebraic difficulties.

Instrumental competence
There are differences concerning this competence. In the first problem Marta does not state any conjecture, whereas in the second problem she tries to conjecture the position of the point P on the median. In this case, the use of GeoGebra has a key role. Moreover, Marta understands the need of proving, as she uses GeoGebra to check her intuitions and conjectures but is aware of the need of proving the conjecture. As stated by Marrades and Gutiérrez (2000), when using DGS “many students begin by using empirical checking and when they have understood the problem and the way to justify the conjecture, they continue by writing a deductive justification. It is also usual to make several jumps among deductive and empirical methods during the solution of a problem” (Marrades & Gutiérrez, 2000, p. 94). In this particular case, the exploration phase does not lead to the construction of a proof, but Marta is aware of the need of proving.

Deductive competence
In the second question of the scaled triangles problem, the visual-algebraic obstacles obstruct the construction of a proof. Marta is not able to relate the trisection of the median with the trisection of the segment BC. The tutor helps Marta to identify the geometric properties of the centroid, and she uses as an accepted theorem that the medians meet in the centroid which trisects each median. Nevertheless, she does not understand adequately the trisection produced, as we can see in her written work and her drawing. She writes ‘tracem paral·leles al costat BC per els punts que parteixen al costat AC en tres parts iguals I coincidiran amb la mitjana AM pel baricentre’. For the first question, she produces a deductive justification, with the help of the tutor (cognitive message of level two). The fact that Marta has more difficulties concerning this problem is also due to the fact that the root problem is a problem of level two. The logical structure and the geometrical resolution approach are simpler for the root problem. In the root problem, the geometrical approach is based on discerning congruent triangles, whereas in this problem the geometrical approach is based on discerning similar triangles.
Feedback
The scaled triangles problem helps Marta to develop the instrumental competence and the structural competence. She understands the logical structure of the problem and the use of GeoGebra helps her to understand the problem as well as to distinguish between the drawing and the figure, in this particular context. This distinction is a necessary step for developing the deductive competence. Nevertheless, Marta has visual-algebraic difficulties that obstruct the resolution process. We consider that we should include problems to foster the visualization.

b) Transitions between the scaled triangles problem and the median problem
In the tables (Tables 4.5 and 4.6), we show the scores referring to Marta’s acquisition degrees of the geometrical competences for the scaled triangles problem, and the median problem. We compare the different acquisition degrees for each competence.

Visualization competence
For the visualization competence, comparing the above tables (table 4.6 and table 4.7) we can observe at a first glance that there are more (+) and (o) scores for the visualization competence for the third problem. In the scaled triangles problem, Marta remained at a graphical-field level and had difficulties in discerning properties of the specific drawing and properties of the geometrical figure. We conjecture that these difficulties are related to Marta’s lack of conceptual knowledge concerning triangles and Thales theorem. We consider conceptual knowledge as knowledge that involves relations or connections (Hiebert & Lefèvre, 1986). For instance, in the second problem she tries to apply blindly Thales theorem considering only the usual one-step configuration in the triangle ABD (Figure 4.63) to compare the ratios AB:EP and BD: ED. She does not consider the global visualization of the triangles ABD and ADC and their common side AD. The instrumentation of the ‘dragging combined with 2-steps visual ratio comparison’ is difficult because it requires a conceptual knowledge of Thales property, which allows the students to extract the proper ratios.
Moreover, she does not connect Thales theorem with similar triangles which are not in the standard configuration, algebraic operations with segments as quantities. This lack of connection makes the visualization of the algebraic relationships between quantities difficult, and thus the visualization of the segment trisection. (-) Difficulty to visualize the relation between the trisection of the segment BM with the trisection of the base BC which is twice the segment BM, Table 4.6.

She does not extract the inner triangle EPF. This may be due to the lack of knowledge about triangle similarity criteria that can be used to justify that the triangles ABC and EPF (Figure 4.63) are similar. We wonder if she connects Thales property with similarity of triangles. As stated by Laborde and Capponi (1994), “les interprétations d’un même dessin en tant que signifiant d’un objet géométrique sont multiples pour deux raisons: la première tient à ce que les interprétations dépendent du lecteur et de ses connaissances, la deuxième tient à la nature même du dessin; à lui seul il ne peut caractériser un objet géométrique” (p. 169).

Instead, in the third problem, she is able to extract two inner triangles, BHM and MFC (Figure 4.60) from the initial configuration, but in this case the triangles are congruent and she has in mind a clear strategy which consists in proving the equality of the heights BH and FC (+) Operative apprehension, Table 4.17.

The use of GGB and the teacher’s orchestration help Marta to visualize the exterior heights of the triangle (-) Difficulty in visualizing exterior heights of a triangle, Table 4.17, and to overcome the initial visualization difficulty. Overcoming this obstacle helps Marta to visualize the inner congruent triangles (Figure 4.60). Nevertheless, she does not extract equivalent triangles (same area), but rather focuses on comparing the heights of both triangles instead of considering the four inner triangles in ABC:

**Structural competence**

We observe relevant differences between table 4.16 and table 4.17. In the second problem, Marta has tries to apply blindly Thales theorem without a clear strategy, whereas in third problem she tries to understand the statement of the problem. The fact that there are no references to parallel lines in the statement of the problem may prevent Marta from considering Thales theorem and then focusing on other properties. As stated
by Lesh (1979), during problem solving, students focus on limited aspects of the problem or misinterpret the problem because the ‘preconceived biases and self orientated perspectives’.

As a result of the previous problem (she is confronted by a similar problem) and the teacher’s orchestration, she understands the motion dependency and there is a shift in the conceptual understanding of the notion of quantifier ( Understanding of quantifier: ‘P any point of the median’). We see here the development of mathematical meaning as an aspect of the orchestrated instrumental genesis. She improves her understanding about the freeness/dependency of GGB objects.

There is a shift with respect to the second problem, as she is able to identify geometric properties and does not consider any more spatial properties as hypothesis. This may be related with the previous observations, and as a result Marta shifts from the drawing to the figure notion, as we can see in these differences. The appropriation of the instrumented scheme ‘dragging to find a counterexample’ is a proof of Marta’s ability to make this distinction (median problem). In the third problem, the mediation of the tutor prompts her to consider the distinction between properties of the drawing and properties of the geometric figure. As stated by Strässer (1992), dragging offers mediation between drawing and figure and can only be used as such at the cost of an explicit introduction (tutor’s orchestration).

For instance, she extracts the two inner triangles BHM and MFC (Figure 4.60) and applies the property of parallel lines intercepted by a secant to justify the congruence of the angles \(<B\) and \(<C\) (Congruence of Alternate angles) instead of considering Thales theorem to justify the congruence of both triangles. This may be due to the difficulty in identifying Thales configuration in this case, and also to the fact that she does not associate Thales theorem with similarity of triangles (factor one in this case). She also applies congruence criteria to both triangles. Despite the fact that there is a shift to a higher degree in this competence, she still has difficulties in considering the notion of equivalent triangles (same area).

**Instrumental competence**

There is also a shift in the instrumental competence to a higher degree competence. In the third problem, Marta understands the need of justifying her conjecture (She understands the necessity of proof). We conjecture that the ability to analyse the drawing in terms of geometrical properties helps Marta to elaborate a deductive proof. She uses the figure and the instrumented schemes ‘dragging combined with measure tools to validate a conjecture’ to validate conjectures and control results. Instead, in the second problem, she is not able to extract geometric properties of the figure and proposes a justification based on using measure tools to justify the equality of segments in one particular case. Nevertheless, she does not consider dragging all the elements of the figure to validate the conjecture (vertices A, B and C for example). The instrumental genesis of this scheme is not evident.

**Deductive competence**

As a result of the transitions to a higher degree in the previous competences, we observe also a higher degree for the deductive competence. In the exploration phase of the third
problem resolution, Marta identifies geometric properties that will lead to the construction of an explanatory proof in the execution phase. There is a shift in the understanding of invariants. In the second problem Marta associated the property ‘D midpoint of the segment EF’ with a particular configuration (E and F trisection points), whereas in the third problem she starts to appropriate the instrumented scheme ‘dragging to find invariants’. She also uses measure tools in both problems, but there is a shift in the way of using them from *wandering measuring* to *validation measuring* (Olivero & Robutti, 2007) in the third problem. These facts help Marta to improve the elaboration of a proof. As stated by Chazan (1990), “*to successfully explore a geometry problem with DGS, students must be able to verify, conjecture, generalize, communicate, prove and make connections*” (Chazan, 1990, p. 530).

Despite the fact that she uses the congruence criterion wrongly to justify the congruence of two triangles, there is a shift in her reasoning (deductive competence), as we can observe comparing Table 4.16 and Table 4.17. In the third problem, the proof is produced with rigorous reasoning, using the congruence of triangles extracted from the initial configuration, whereas in the second problem she has difficulties of justification. We conjecture that the fact of understanding the motion dependency in terms of logic dependency and the appropriation of the schemes ‘dragging combined with measure tools to validate a conjecture’, ‘interpreting round-off error to validate a property of a figure’ also foster deductive reasoning in the resolution of the third problem. The tutor’s orchestration is relevant for the appropriation of the instrumentation scheme ‘dragging to find a counterexample’, which has been not considered in the second problem. As a result of a tutor’s validation message, Marta considers for the first time dragging the vertices to find a counter-example.

**Feedback**

We consider that the visualization of the algebraic relationships between quantities as segments is still an obstacle for Marta. For instance, she only observes empirically, for one concrete triangle, that the centroid trisects the median. She may need more practice on this issue. She also has difficulties in connecting Thales property with similar triangles. We may introduce problems to identify similar triangles.

We conjecture that the insertion of the figure of the second problem in a grid may obstruct the process of validating conjectures in other triangles. Marta starts to validate properties in other triangles through dragging, but she drags vertices in a limited area and fixes all the figures in the origin of coordinates. The tutor should focus on the instrumentation of the schemes ‘dragging to validate the construction’ and ‘dragging to validate a conjecture’ during the orchestration ‘on the spot’ (Drijvers et al., 2009). There is also a question of didactical contract and ‘usage schemes’. We consider that GGB can be used to help students build mental constructs that are useful skills and prerequisite skills for analytical thinking.

Marta does not consider strategies based on comparing areas of figures applying equivalence of figures and area properties. We may propose problems that foster these kinds of strategies, or consider the possibility of giving cognitive messages of higher level to give insight into this strategy. The use of different strategies may result in a richer concept of distance and areas of plane figures.
c) Transitions between the median problem and the quadrilateral problem
In the tables (Tables 4.6 and Table 4.7) we show the scores referring to the acquisition degrees of the geometrical competences for the median problem and the quadrilateral problem. We compare the different acquisition degrees for each competence.

Visualization competence
As in the third problem, Marta constructs again an isosceles triangle as a general case (o) score in Table 4.6 and Table 4.7), but she is aware of the distinction between figure and drawing ((+) Distinction between drawing and figure). She states ‘No però mira això, clar però aquí no és exacte’ (lines 28-31). She considers that a paper-and-pencil drawing is not precise. She may refer to the difference with working with GGB where she can ‘trust’ properties of the drawing. When working with GGB, Marta uses measure tools combined with dragging to validate conjectures (instrumented scheme). Nevertheless, in the third problem, she is not aware of the necessity of considering other triangles to generate a large number of examples. She starts to drag elements influenced by the tutor’s orchestration. Instead, in the fourth problem she considers particular cases (right-angled triangles) as a tool for visualizing properties and is aware of the need to generalize to other triangles, as these are particular cases (Table 4.7, (+) Distinction of a particular case of a right-angled triangle from the general case).
Nevertheless, she still focuses on comparing the elements of triangles (base-height) to compare areas but does not try to compare areas directly. For instance, through particularization she obtains the following figure (Figure 4.65) but does not relate it with the previous problem (the median AN splits the triangle into two triangles of equal area). This may be due to the fact that she does not have the property of the median in mind, as we have already observed in the resolution of the third problem. It also may be due to a visual-algebraic obstacle to understanding the equivalence between the area of the triangle ANC and the sum of the areas of the triangles AMP and PNC.
She focuses on splitting the quadrilateral BMPN into two triangles to obtain its area using the formula for the triangle’s area. She tries to obtain relations between the heights of these triangles through perceptual approach. She is misled by a specific property of isosceles triangles (o) Property of an isosceles triangle through perceptual approach). The tutor proposes that she finds a counterexample.

Figure 4.65: particularization of the triangle and the point P

There is a shift in the visualization competence as she considers changing the initial triangle ABC. She also considers particular positions of the point P (in the root problem, which had to be solved with paper-and-pencil she reasoned on the static figure). Nevertheless, she does not consider continuous variation of the point P; instead she focuses on particular cases, as we can see in her statements and graphic actions. We remark that in the GGB resolution of the third problem she uses photo-dragging (Olivero, 2002) to obtain also particular configurations.
Another relevant property is that she extracts two similar triangles (ʻ+) She extracts two similar triangles, operative apprehension). Nevertheless, she does not visualize the outside triangle ABC to compare areas, and she focuses on the inside quadrilateral.

**Structural competence**

There is a shift in the structural competence. She connects Thales theorem with the concept of proportionality and similar triangles. The tutor’s orchestration helps Marta to understand the reciprocal property. She was used to applying Thales theorem only to work out the length of given segment but she still does not connect Thales theorem with areas of triangles.

She understands the logical structure of the problem (quantifiers: any triangle ABC, any point P) despite the fact that she still considers an isosceles triangle. For instance, she starts to consider other triangles and tries to reason on geometric properties of the figure.

**Instrumental competence**

We observe relevant differences between Table 4.6 and Table 4.7. This is due to the fact that during the resolution of the fourth problem she does not mention the necessity of justifying the observed properties. Nevertheless she is aware of the necessity of proving, as we have shown in the third problem.

These differences between both problems may be due to two factors. On the one hand, she has difficulties in justifying deductively that the lines MN and AC are parallel (midpoint theorem), but she is perceptually convinced of the truth of the statement (parallelism of both lines). In this case, the tutor’s orchestration fosters the awareness of the need of proving.

On the other hand, the tutor guides the resolution process of the fourth problem and remarks on the necessity of justifying the geometric properties.

**Deductive competence**

There are no relevant differences for acquisition degree of the deductive competence. In the fourth problem, Marta’s strategy is based on obtaining both areas using algebraic expressions. She justifies deductively the relation between the elements of the triangles using Thales theorem, but she has difficulties in justifying that the lines (MN) and (AC) are parallel. The tutor’s orchestration has a key role in giving insight for the proof of the midpoint theorem. Marta focuses on comparing measures of sides and heights rather than looking for areas directly in both problems.

**Feedback**

We have shown that there is a shift in the visualization and structural competence. Marta understands the logical structure of the problem and considers varying the initial triangle. Moreover, she is triggered to think about other aspects of Thales theorem.

Nevertheless, she still has visual-algebraic obstacles. For instance, she does not observes the relation between the fact that if the areas of the quadrilateral and the sum of the two inner triangles are equal, then the area of each one is half the area of the outside triangle ABC.

We also remark that she considers similar triangles but does not mention the ratio of its areas. Moreover, she uses resolution strategies based on comparing linear elements of
triangles (base-height) and applying the area formula of a triangle. She does not consider strategies based on comparing areas directly. We propose considering problems to foster this kind of strategies.

We also propose to generalize the fourth problem to foster the understanding of Thales reciprocal in a more general case. For instance, we may consider other ratios. We propose the following generalization:

Let ABC a triangle and let P any point of the side BC, M and N be the points of the sides AB and AC, respectively, such that the segments BM, MA and CN, NA have the ratio 1: k.

What relation is there between the area of the quadrilateral MPNB and the addition of the areas of the triangles AMP and PNC?

![Generalization of the quadrilateral problem](image)

**Figure 4.66: generalization of the quadrilateral problem**

In this case, \((AMP) + (PNC) = \frac{1}{k} (ABC)\)

\((BMPN) = (ABC) - \frac{1}{k} (ABC) = \frac{k-1}{k} (ABC)\)

Thus the ratio between both areas is \((k-1)\).

### 4.2.3.2 Transitions concerning Guillem’s local HLT

In this section we reflect on the transitions concerning Guillem’s learning trajectory.

**a) Transitions between the root problem and the scaled triangles problem**

In the tables (Tables 4.2 and 4.8) we show the scores referring to the acquisition degrees of the geometrical competences for the root problem and the scaled triangles problem. We compare the different acquisition degrees for each competence:

**Visualization competence**

There are (-) scores concerning the visualization competence in both problems. In the root problem, Guillem has difficulties in visualizing the inner congruent triangles. This may be due to several factors. Firstly, in the root problem, Guillem does not understand that E is any point of the diagonal. At first, he focuses on applying trigonometry of the right-angled triangle to find the concrete lengths of the segments and then apply the area formula of the rectangles. With the help of the tutor, he focuses on a resolution approach based on equicomplementary dissection rules, but finally he uses the area formula of a triangle to validate with algebraic resolution the equality of areas. Secondly, Guillem...
does not tend to base his reasoning on figures. He is not a ‘geometric’ (Krutetskii, 1976) student. Instead, in the second problem, Guillem understands the logical structure of the problem but has difficulties in visualizing the exterior height of a triangle. In this particular case, this is also due to an inadequate understanding of the height’ definition. Nevertheless, Guillem visualizes three similar right-angled triangles in the root problem and in the second problem he constructs and auxiliary element (another median) and visualizes the trisection of the triangle’s base considering parallel lines (mental operative apprehension).

Structural competence
There is a relevant shift in the structural competence, which can be observed in both tables (one (-) score in table 4.2, one (+) and one (o) score in table 4.8). In the root problem, Guillem had difficulties understanding the statement of the problem. He had difficulties with the quantifier ‘E any point of the diagonal’. He also had difficulties in interpreting the solutions of a linear system of equations. Instead, in the second problem, he understands the logical structure of the problem which is more complex. The tutor’s orchestration in the first problem and the proposed tasks (first and second problem) help Guillem to understand it. Also the use of GeoGebra has a key role. Guillem understands, with the help of the tutor, the relation between free/dependent GGB objects and the concept of quantifier.

Instrumental competence
There is a shift in the structural competence. In the first problem, Guillem tries to obtain the areas of both rectangles and does not try to state a conjecture. Instead, in the scaled triangles problem by dragging the point P along the median he validates the equality of segments and tries to state a conjecture about the position of the point P. In this case, the use of GeoGebra has a key role.

Deductive competence
In the root problem, Guillem has difficulties in elaborating a deductive proof. This may be due to the fact that he has difficulties understanding the structure of the problem. He applies equicomplementary dissection rules to justify the equality of areas but is not able to justify deductively his reasoning process. He expresses the equality of areas using the area formula of a triangle. He obtains equivalence, but at first he tries to solve it and he obtains the tautology 0=0. He has difficulties in interpreting it, but finally he understands that it is equivalent to the equality of areas.
Instead, in the second problem he justifies deductively that the centroid is the point that verifies the required property. Nevertheless, he has to accept the properties of the centroid as a key-lemma not proved.

Feedback
In the resolution of the second problem, Guillem overcomes the difficulties in visualizing the exterior heights of a triangle. He had difficulties with the height definition, as he considered it as a perpendicular line to the base without considering the vertex. The tutor’s messages and the use of GeoGebra have a key role in making Guillem understand
his misconception. Nevertheless, Guillem still has difficulties with the transfer of algebraic number theoretic properties to the arithmetic of geometric quantities. Moreover, in the resolution of the second problem, Guillem does not drag the vertices of the initial triangle.

b) Transitions between the scaled triangles problem and the median problem

In the tables (Table 4.8 and 4.9) we show the scores referring to the acquisition degrees of the geometrical competences for the scaled triangles problem and for the median problem. We compare the different acquisition degrees for each competence.

Visualization competence

There is a similar degree of visualization for both problems, as we can observe in tables 4.8 and 4.9. In the scaled triangles problem Guillem has difficulties with the algebraic operations with segments as quantities, but he is able to connect the trisection of a segment with Thales property. He avoids the previous obstacle through operative apprehension, introducing another median of the triangle. In the third problem, the use of GGB and the tutor’s orchestration helps Guillem to visualize the exterior heights of a triangle. Guillem has the notion of the height as a perpendicular line to the common base, but he links this line to the point P, and thus to drag the perpendicular line along the median he has to modify the triangle, as the line is a dependent object (Figure 4.67). We conjecture that this is a technical obstacle. Guillem was trying to construct a free perpendicular line through a point of the median. Guillem avoids overcrowding constructions and the use of measure tools. It may be due to the fact that he has not appropriated the usage scheme ‘hide and show objects’. He also avoids dragging the vertices of the triangle. We conjecture that he has not appropriated the scheme ‘dragging to validate the construction’ and is afraid that changes may ‘mess-up’ the figure. As stated by Laborde, Kynigos, Hollebrands and Strässer (2006), “it has been observed that when students use the drag mode, they do not use it in a wide zone but on a small surface as if they were afraid to destroy their construction” (Laborde et al., 2006, p.286). Nevertheless, he is aware of the fact that he has to use geometric properties to construct a robust figure. Guillem avoids dragging elements in a wide range and overloading the construction with elements.

![Figure 4.67: Wrong construction of the height (Guillem)](image)

The immediate perceptual approach is an obstacle in the third problem, but Guillem understands the distinction between drawing and figure, and the use of GGB helps him in
the process of visualization (exterior heights of a triangle, extracting congruent triangles, identifying geometric properties).

**Structural competence**
There is a shift in the structural competence degree of acquisition, as we can observe in tables 4.8 and 4.9. Guillem understands in the third problem the motion dependency in terms of logical dependency within the geometrical context. The tutor can exploit Guillem’s experience in the third problem to attach mathematical meaning to a technical obstacle (related to the dragging tool) encountered by Guillem. He wrongly selects a point close to the point P and drags the polygon APC instead of the point P. He has difficulties understanding the motion dependency of this object and its elements of lower dimension.

The acquisition of the instrumented scheme *dragging to find invariants* helps Guillem to state conjectures. For instance, in the third problem he conjectures the equality of heights by dragging continuously the point P along the median and observing invariant elements. Nevertheless, he does not consider dragging all the elements of the figure. Thus he does not validate his conjecture in other triangles. Moreover, he considers an isosceles triangle.

**Instrumental competence**
There are no relevant differences in the instrumental competence in terms of scores. Guillem understands the need of proving despite the fact that in the second problem he has difficulties constructing a proof. In the second problem the exploration phase does not lead to the construction of a proof and the tutor helps Guillem to find geometric properties that lead to the construction of a proof based on the accepted fact (justified empirically) that the centroid trisects the medians, whereas in the third problem the exploration phase leads to the construction of an explanatory proof.

Guillem understands that the exploration through dragging is not sufficient to guarantee the truth of the observations made for both problems, and this reveals insight into the competence.

**Deductive competence**
There is a shift in the acquisition degree of the deductive competence. In the third problem, Guillem produces a proof with deductive reasoning, whereas in the second problem he had difficulties constructing a proof. Nevertheless, he still considers particular triangles (isosceles triangle), and in the familiarization phase of the third problem he states ‘com que no fa referència a dades concretes puc considerar un triangle rectangle’. He does not consider that the properties used may be dependent of the particular triangle constructed. Despite this fact, the properties used are independent of the concrete triangle considered by Guillem. We consider that the principal obstacle to be overcome is the lack of awareness of the general character of a free object. For instance, Guillem does not understand that given a triangle, the object ‘triangle’ is quantified universally and he should consider, with GGB, a larger family of triangles. This obstacle is not provided by GGB. It can be considered as an already existing obstacle that becomes manifest by working with GGB (Drijvers, 2002).
Feedback
We conjecture that the insertion of the figure of the second problem in a grid may obstruct the process of validating conjectures in other triangles. Guillem understands the notion of invariant and appropriates partially the scheme ‘dragging to find invariants’, but he does not consider dragging the initial triangle. Moreover, he tends to consider particular cases, such as right-angled triangles, isosceles triangles as a general case.
The tutor should focus on the instrumentation of the schemes ‘dragging to validate the construction’ and ‘dragging to validate a conjecture’ during the orchestration ‘on the spot’ (Drijvers et al., 2009). There is also a question of didactical contract and ‘usage schemes’. The use of GGB, orchestrated by a tutor, can be used to help students build mental constructs that are useful skills and prerequisite skills for analytical thinking.
Guillem does not consider strategies based on comparing areas of figures applying equivalence of figures and area properties. We may propose problems that foster this kind of strategies, or consider the possibility of giving cognitive messages of higher level to give insight in this strategy. The use of different strategies may foster a richer concept of distance and areas of plane figures.
We should change the statement of the problems to highlight the aspect of ‘universal quantifier’. If we do not insert a figure in the statement of the problem, some students consider only particular cases. We may include figures, without a grid, in the statement of the problem to foster the consideration of general cases, and also consider cognitive messages to foster this distinction.

c) Transitions between the median problem and the quadrilateral problem
In the tables (Tables 4.9 and 4.10) we show the scores referring to the acquisition degrees of the geometrical competences for the median problem and the quadrilateral problem. We compare the different acquisition degrees for each competence.

Visualization competence
There are no relevant differences concerning the scores between both tables (Tables 4.9 and 4.10). Nevertheless, Guillem visualizes the dynamic variation of the point P in the quadrilateral problem, which as to be solved with paper-and-pencil. We consider that this fact reveals a shift in the visualization competence. Instead, in the median problem the dragging tool helps Guillem to visualize continuous variations of the point P along the median. Guillem has difficulties in visualizing Thales configuration (non standard positions) in the fourth problem, as we have already noticed in previous problems. Finally, with the help of the tutor, he overcomes these difficulties, and through operative apprehension he considers an auxiliary parallel line to apply Thales theorem.
We observe again algebraic-visual difficulties. Guillem does not visualize that if both areas are equal then each one is half the area of the outside triangle.

Structural competence
There is a shift concerning the structural competence. Guillem understands the logical structure of the problem and identifies that the heights of all the inner triangles have the same length. In the quadrilateral problem she identifies that both areas are constant. In the previous problem, Guillem had difficulties with the definition of the heights, whereas in this problem he shows an adequate understanding of the height definition. He also
connects the concept of height with the concept of distance from a point to a line, as we can see in the graphic representations made with paper-and-pencil. In the median problem, he had technical difficulties related to an inadequate understanding of the definition of height (perpendicular line to the base). We notice that the software does not help the students to overcome this difficulty as it allows the intersection of a segment and a line to be constructed.

**Instrumental competence**

The most relevant difference is that Guillem understands the necessity of considering a generic triangle. For the first time, Guillem does not try to reason only on particular triangles. Moreover, he is aware of the distinction between drawing and figure. For instance, in the quadrilateral problem he tries to validate his conjecture (parallelism) in several triangles, and he considers also degenerate cases. Nevertheless, when using GeoGebra, he checks the parallelism of two segments \([PQ]\) and \([RS]\) by constructing a parallel line to the segment \([PQ]\) through the point \(R\). As the point \(R\) belongs to the constructed line, he accepts the parallelism as a true statement. This is related to the difficulty to prove Thales reciprocal.

**Deductive competence**

There are more (+) scores for this competence concerning the resolution process of the median problem. Nevertheless, this is due to the fact that, in the quadrilateral problem, Guillem has difficulties in proving Thales reciprocal. This is due to the fact that it requires the understanding of the proof by contradiction. We do not take into account this fact, as we do not expect from these students the understanding of the proof by contradiction.

**Feedback**

Guillem has improved his geometric competences. Nevertheless, he still has some difficulties with concepts and procedures related to these problems. For instance, he still has difficulties in visualizing algebraic relationships between quantities as segments. We also observe that he does not connect different resolution strategies and tends to use resolution approaches based on obtaining equality of ratios.

### 4.2.3.3 Transitions concerning Aleix’s local HLT

In this section we reflect on the transitions concerning Aleix’s learning trajectory.

**a) Transitions between the root problem and the scaled triangles problem**

In the tables (Tables 4.3 and 4.11) we show the scores referring to the acquisition degrees of the geometrical competences for the root problem and the scaled triangles problem. We compare the different acquisition degrees for each competence.

**Visualization competence**

There is a shift in the visualization competence. Comparing the above tables, the number of (+) scores is similar, but in the second problem Aleix visualizes algebraic-geometric relationships of segments, which is one of the purposes of the second problem. The use of GeoGebra has a key role in the visualization process. For both problems, Aleix has the ability to discern similar and congruent triangles. Aleix is a ‘geometric student’ and he
likes to think in terms of figures. He tends to avoid algebraic approaches for the resolution of both problems. A relevant aspect is that he does not pay attention to the concrete data of the root problem. He discerns that the reasoning based on applying equicomplementary rules holds for any rectangle and any point P of the diagonal. In the second problem, he does pay attention to the grid in which we insert the figure of the statement of the problem. Aleix is the only student who constructs with GeoGebra, in the familiarization phase, a different triangle. The visualization competence is high for both problems.

**Structural competence**

There are no relevant differences for the structural competence comparing the (+) and (+++) scores in both tables. Nevertheless, the level of difficulty of the second problem is higher, and we can interpret that there is a shift in the structural competence. Aleix understands the logical structure of both problems, and in the second problem he starts to gain insight into the geometric transformations. This fact may be due to the mediation of the software for the resolution of this particular task. Also, in the first problem, tutor intervention is relevant to foster Aleix’s awareness of the similarity criteria.

**Instrumental competence**

There are differences for the instrumental competence (one (+) in table 4.3 and one (+++) score in table 4.11). For both problems, Aleix states conjectures and tries to verify these conjectures deductively. In the second problem, we have evidences of Aleix’s internalization of dragging as theoretical control. The higher difficulty of the task and the use of GeoGebra foster the use of the dragging tool.

**Deductive competence**

There are no relevant differences for the deductive competence. For both problems, Aleix justifies deductively the conjectures and shows understanding of the function of definitions and proofs. The only difference is that in the first problem Aleix does elaborate a minimal proof. He is not aware, in the root problem, of the fact that this is not necessary to prove that the inner rectangles are similar to the outside rectangle.

**Feedback**

The resolution of this problem, combined with the use of GeoGebra, helps Aleix to visualize algebraic relationships between quantities as segments. Also the meaning of homothetic transformation starts to come across. The tutor’s intervention in this problem is based on messages of level one to foster Aleix’s autonomy and guide the student in the instrumentation process. The task is not difficult for Aleix, and it is not necessary to give messages of higher level. It may have been better if the tutor had fostered the use of the dilatation tool in the verification phase to let the meaning of homothetic transformation come across. We may also consider introducing new questions in the statement of the problem, such as questions concerning locus, for students who solve the first questions.

**b) Transitions between the scaled triangles problem and the median problem**
In the tables (Tables 4.11 and 4.12) we show the scores referring to the acquisition degrees of the geometrical competences for the scaled triangles problem and median problem. We compare the different acquisition degrees for each competence.

**Visualization competence**

There is a shift in the visualization competence, as we can see in the above tables. Aleix is a geometric student who likes to think in terms of figures and he has the ability to discern geometric configurations. Nevertheless, in the second problem Aleix has to discern similar figures, as in the first problem, whereas in the third problem he does not consider equivalent figures. In this third problem, the use of GeoGebra has a key role in helping Aleix in the visualization process.

We assigned a (o) score in the third problem concerning the visualization of the exterior heights of the triangles. Aleix focuses on justifying deductively the equality of heights, but finally he abandons his attempt. This is due to a lack of visualization concerning the exterior heights of the triangle. We conjecture that, in this case, there are several factors that obstruct the visualization. Firstly, Aleix has used the options grid and axes, and the figure is overcrowded with lines that obstruct the visualization process. Secondly, Aleix does not construct the feet of the heights, and this fact obstructs the visualization process and the strategy based on extracting two congruent right-angled triangles.

**Structural competence**

Aleix understands the logical structure of the problems. There are no relevant differences concerning this competence. As mentioned before, in the third problem he has difficulties discerning the figure and the drawing, but this is due to the fact that he is convinced of the truth of a conjecture, which turns out to be false. Nevertheless, Aleix understands the distinction between drawing and figure. In both problems, Aleix connects different resolution approaches and understands the concepts involved. For instance, in the second problem, Aleix uses the algebraic window and the geometric window simultaneously.

**Instrumental competence**

In both problems, Aleix states conjectures and tries to justify them deductively. He uses the drag tool as a theoretical control and is the only student who drags the elements in a wide area and considers a large family of examples. For instance, he considers degenerate case, he changes the orientation of the figures and searches for invariants algebraic-geometric. We have noticed the use of cinema-dragging while looking for invariants in the algebraic window. For instance, in the second problem he refutes a construction as the result of noticing that the radius of a circumference stays invariant while dragging one of its points. He uses the drag tool to validate conjectures and to find counterexamples, but he is aware of the need of proving.

In the third problem, the use of GeoGebra helps Aleix to elaborate an explanatory proof based on applying the properties of the area function as a measure function.

**Deductive competence**

There are no relevant differences concerning the deductive competence. We have coded with a (++) score the proof elaborated in the third problem because Aleix elaborates a rigorously formulated proof with paper-and-pencil. Nevertheless, the tutor helps Aleix
with cognitive messages of level two, whereas in the second problem Aleix works on his own and the tutor only gives messages of level one. In both cases, the use of GeoGebra helps to coordinate the visualization processes

c) Transitions between the median problem and the quadrilateral problem
In the tables (Tables 4.12 and 4.13) we show the scores referring to the acquisition degrees of the geometrical competences median problem and for the quadrilateral problem. We compare the different acquisition degrees for each competence.

Visualization competence
There is a shift in the visualization competence. This may be due to the fact that in the first two problems the figures can be split into similar or congruent triangles, whereas in the third problem, to follow the strategy ‘equicomplementary dissection rules of areas’ the student has to discern equivalent figures (same area). At the beginning of the third problem, Aleix is convinced of the fact that the inner triangles are congruent and tries directly to justify it deductively, without validating the conjecture. In this case, this behavior is not due to an inadequate insight into the instrumental competence. This fact obstructs the visualization of the inner equivalent triangles. The emergence of instrumented schemes such as ‘dragging to identify invariants of the figure’ fosters the visualization of inner equivalent triangles. Instead, in the fourth problem, Aleix is aware of the strategies based on discerning equivalent figures. He is also aware of reconfigurative operative apprehension, as is used to dragging elements and transforming figures in the previous problems solved in a technological environment.

Structural competence
There are no relevant differences concerning this competence. As we mentioned in the previous section, the (o) score in the third problem is due to the fact that Aleix is convinced of the truth of a false conjecture, and he tries to justify deductively the conjecture. He spends a great amount of time trying to justify deductively the conjecture and at first does not react to the tutor’s messages. Finally, he reacts and refutes the conjecture. During the resolution of the problems, Aleix works autonomously and he wants to solve the problems on his own.

Instrumental competence
There are differences in the instrumental competence. Nevertheless, Aleix’s actions show insight into the competence for both problems. In the third problem there are more (+) scores because the resolution process is richer. Aleix tries several resolution strategies in the third problem, whereas in the fourth problem he focuses on a unique strategy. We have more feedback on the third problem.

Deductive competence
For both problems, Aleix elaborates deductive proofs and there is insight of high degree of acquisition. In the fourth problem, Aleix requires the help of the tutor to prove Thales reciprocal, but as we mentioned before we do not expect from students of this age to be aware of such strategies of proving. Moreover, Aleix develops partial awareness of the proof by contradiction.
For all the students there is an improvement in the visualization, structural and instrumental competence. We do have evidence of improvement concerning the deductive competence in the learning trajectories of Marta and Guillem. In the following chapter we interpret the results of the teaching experiment considering the theoretical framework.
5. Interpretation of students’ solutions through the framework

In this section we interpret the results of the teaching experiment in terms of the theoretical framework. We characterize three prototypic learning trajectories, defined by the three-fold orchestration. We characterize the learning trajectories in terms of the transitions between the micro-cycles and the effect it has on the student, the role of the tutor’s orchestration and synergy of environments.

We assume that the students have the necessary knowledge related to the contents and procedures to solve the proposed problems. The hypothetical learning trajectory (local cycle) it at the starting level, and its role is to extend students’ geometric competences in the context of problems that compare areas and distances of plane figures.

We have identified the following prototypic behaviours: not confident, confident and autonomous. In the following sections we characterize the learning trajectories for each prototypic case. Finally, we compare and discuss the results concerning these thee cases.

5.1 Results on the HLT, the orchestration and the synergy of environments

We formulate the results on the HLT, the orchestration and the synergy of environments for each prototypic case (not confident, confident, and autonomous students). For the sake of clarity, we show again the definitions assigned to each code concerning the four competences. We use these codes in the following sections, to condense in a table the interpretation of the transitions concerning each competence between pairs of problems (e.g. Tables 5.1, 5.2, 5.3 and 5.4 in the case of non confident students). The (++) scores refer to observations that reveal a deep insight into the competence (unexpected, original); the (+) scores refer to observations that reveal the expected insight into the competence; the (o) scores refer to insights into the competence that are not completely correct but do contain valuable elements or observations done with the help of the tutor, and the (-) scores refer to conceptions concerning the competence that are not adequate.

For each competence (visual, structural, instrumental and deductive), we define more precisely the meaning of the four scores, and we give indicators for each score and competence (see chapter 3. Research design and methodology).

5.1.1 Results on the Hypothetical Learning Trajectory (HLT)

In this section we formulate the feed-forward of the local cycle and report the transitions between the micro-cycles, concerning each competence for each prototypic case (not confident, confident, and autonomous).

5.1.1.1 Results on the HLT: ‘Not confident’ profile

These students are high-achieving students who have a harmonic profile (Kruteskii, 1976). They are not confident in their mathematical skills or their ability to solve the proposed problems, and they trust the software measurements. In this section we formulate the feed-forward of the local cycle and report the transitions between the micro-cycles concerning each competence.

During the teaching experiment, there are no relevant changes in these students’ cognitive structures concerning the procedures used to solve the problems. They focus on resolution approaches based on obtaining equality of ratios to compare linear elements,
and then they apply the area formula. In the root problem, the tutor’s intervention helps them to discern congruent triangles and to focus on a strategy based on equicomplementary dissection rules. Nevertheless, in the following problems they do not apply strategies based on applying equicomplementary dissection rules. This is due to several factors. Firstly, it may be due to visualization difficulties in discerning equivalent or similar triangles in non-standard positions. Secondly, it may be due to the fact that students have a priori conceptions about the procedures used to solve these problems and they lack confidence to try other procedures. For instance, Marta associates these problems to the use of Thales theorem and does not consider other strategies. Marta states ‘triangles és Thales o Pitàgores’.

As stated by Lesh (1979), during problem solving students may focus on limited aspects of the problem or misinterpret the problem because of the preconceived biases and self-orientated perspectives. During the teaching experiment they consider applying equicomplementary dissection rules, but only in the particular case of congruent figures (for instance extracting congruent triangles in the root problem and considering particular cases for the other problems).

The use of GeoGebra does not lead these students to consider strategies based on equicomplementary dissection rules. They combine different resolution approaches, based on the use of figures and algebraic resolutions. We identify the transition to a higher visualization acquisition degree and also for the structural and instrumental competences. Instead, there are no relevant changes in the deductive competence.

In the following sections, we identify the elements that facilitate these transitions for each competence and the elements that obstruct the transitions.

The visualization competence

There is a shift in the visualization competence in this teaching experiment. The students overcome visualization difficulties, but they still have visual-algebraic obstacles. At the start of the first micro-cycle (root problem), the students do not visualize other positions of the point E on the diagonal and have difficulties conjecturing the equality of areas. The instrumental genesis of the scheme ‘dragging to validate a conjecture’ and ‘dragging to find invariants’ helps the students to understand the dynamic structure of these problems during the teaching experiment. Nevertheless, they have difficulties with this scheme, as they do not identify all the elements to be dragged. For instance, in the second problem, Marta identifies that the centroid is the point which verifies the required property, but she has difficulties identifying its geometric properties, namely that the centroid trisects the median. As she does not drag the vertices of the triangle and has difficulties interpreting the round-off error, she does not discern the centroid trisection property. Marta uses measure tools to validate her conjectures and trusts that measurements with GeoGebra are exact. In this case, the tutor helps Marta to interpret the round-off error.

The students start progressively to consider varying the initial triangle and the variable elements during the teaching experiment. For instance, in the problem of the median Marta drags the initial triangle to consider an isosceles triangle. She states ‘la mitjana parteix el triangle per la meitat però això depèn del triangle’. In the quadrilateral problem, which has to be solved with paper-and-pencil, Marta considers other positions
of the point P and other triangles, and she uses particular cases to visualize the geometric properties. Nevertheless, she does not visualize continuous variations either with GeoGebra (discrete dragging, Olivero (2002)) or with paper-and-pencil.

In the quadrilateral problem, Marta draws different triangles to visualize that the heights of the triangles ANM and MNP (Figure 5.1) are equal. In this case, she discerns two inner equivalent triangles in the quadrilateral BMPN, but she has more difficulties visualizing that the heights are equal in a general case (see figure 5.1c). Finally, she overcomes these difficulties and draws an auxiliary parallel line.

![Fig. 5.1a: Particular right-angled triangle](image)

![Fig. 5.1b: Particular position of the point P](image)

![Fig. 5.1c General triangle and general point P](image)

![Fig. 5.1d: Triangles in standard Thales configuration (2MN=AC)](image)

**Figure 5.1:** Particularizations of the triangle (right-angled triangle) and the point P on the base AC

Finally, as mentioned before, we conclude that the students have still visual-algebraic difficulties. This is the case of Marta, who in the scaled triangles problem does not visualize the trisection of a segment in a non-standard position. In the quadrilateral problem (Figure 5.1), she does not visualize the relation between the areas of both triangles and the quadrilateral, as we can see in the paragraph below:

Marta: la resta entre una àrea [MPNB] i l’altra [triangles AMP and PNC] sempre serà constant.
Els altres dos [both triangles AMP and PNC] ja he vist que és constant
Tutor: Però quina relació tenen?
Marta: Que la resta és constant. Això no és una relació?
Tutor: Sí, però quina constant?

We see in the above paragraph that Marta does not visualize that the addition of areas is constant (area of the outside triangle). Instead, she states that the subtraction of areas is constant and does not identify that both areas are equal.

Another relevant shift is that the students develop a deeper understanding of the distinction between figure and drawing. The instrumental genesis of dragging to find a counterexample (median problem) involves the awareness of the distinction between drawing and figure. At the start, they had difficulties in discerning geometric properties from properties of the drawing. The tutor’s orchestration, the use of GeoGebra and the proposed tasks help the students to overcome these difficulties.

In the following table (Table 5.1) we report the different codes concerning the visualization competence for each micro-cycle.
Table 5.1: Scores concerning the visualization competence (Marta)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning visualization competence</th>
</tr>
</thead>
</table>
| Root problem (P&P) | (+) She does not visualize either the variation of the point P along the diagonal or the particular case mentioned  
(-) Difficulty in visualizing the equality of areas  
(+) She discerns inner congruent triangles with the help of the tutor |
| Scaled triangles problem (GGB) | (o) Distinction between drawing and figure (lines 5-10 and lines 42-45)  
(o) Visualization of the point P in the median but not the ratio 1:3  
(-) Difficulty in visualizing the relation between the trisection of the segment NM with the trisection of the base (BC)  
(-) Difficulty in visualizing the relation between the ratio 1:2 and the ratio 1:3 in the median |
| Median problem (GGB) | (-) Difficulty to visualize the exterior heights of a triangle  
(+ ) Operative apprehension: she extracts two congruent right-angled triangles with the help of the tutor  
(+ ) Distinction between drawing and figure (she visualizes that the median splits the triangle into congruent triangles only for isosceles triangles) |
| Quadrilateral problem (P&P) | (+) Visualization of variations of the point P (visualizes particular cases)  
(+ ) Operative apprehension: she discerns two equivalent triangles inside the quadrilateral  
(+ ) Difficulty in visualizing that the addition of areas is constant (algebraic-visual distinction)  
(+ ) Difficulty in visualizing Thales configuration in a non-standard situation (auxiliary parallel line required)  
(+ ) Distinction between drawing and figure (she discerns geometric properties) |

The structural competence

The instructional design based on similar problems helps the students to understand the logical structure of the problems. The role of GeoGebra and the tutor’s orchestration also elicits this understanding. There is a shift in the structural competence concerning the understanding of the logical structure of the problem. In the root problem, some students consider the length AE as an unknown (E any point of the diagonal AC), but they have difficulties to consider it as a parameter. This leads the students to the strategy based on trying to work out the concrete lengths of the sides of the rectangles whose areas have to be compared. With the help of the tutor, Marta finally conjectures the equality of areas and she expresses a solution based on equicomplementary dissection rules in natural language. Nevertheless, when she tries to transfer her thought to words on the worksheet she is not successful and gets lost (she tries to shift to the algebraic register and she changes completely the resolution strategy). This could be due to a lack of skills to transfer mental reasoning into a written deductive proof.

The scaled triangles problem has a key role for the understanding of the logical structure of the subsequent problems. The constructive flavour of the problem (existential quantifier\textsuperscript{12}) and the tutor’s orchestration help the students to understand the logical structure of the problem and the distinction between drawing and figure. For instance, in the case of Marta we can see in the paragraph below that she overcomes partially the initial difficulties.

5. Marta: E i F són punts mitjors [Figure 5.2]  
6. Tutor: No ho diu l’enunciat  
7. Marta: Aquí sí que està al mig…Però no té perquè ser així…  
8. Marta: Però és que aquí sí que està al mig…Es veu en el dibuix que E i F són punts mitjors.  
9. Tutor: no ho saps

\textsuperscript{12} That is a constructive existential quantifier
10. Marta: Ah, no! Mira. On he de situar P per obtenir BE = EF = FC. Llavors no pot estar al punt mig… [she reads again the statement of the problem]

11. Marta: Aquestes paral·leles han d’estar més avall. He de fer la mateixa figura pels dos apartats [She considers the same position of the point P for both questions: unique point P that verifies EM=MF and trisection of the segment BC]

12. Marta: Divideixo en tres parts… [the base BC of the triangle] i ja sé on està P. [heuristic strategy: considering the problem solved].

Figure 5.2: figure inserted in the statement of the scaled triangles problem

The constructive nature of this problem, the tutor’s orchestration and the synergy of environments encourage also the students to use the drag tool. In this problem, Marta understands the universal condition of the point P. Until this moment she has considered the point P as a concrete point that ‘by accident’ satisfies that EM=MF.

25. Marta: ¿Per i.e. quan aquest és 2 [BE] l’altre no és 2 [EF=1.99]? No es pot fer que això [EF] quedí fix? [dragging for adjusting and round-off error obstacle]

26. Tutor: Tracta de veure primer la relació entre EM i MF. Hauries de fer primer l’apartat a)

27. Marta: Són iguals… [She constructs with the tool intersection the points E and F and she uses the tool distance applied to two points to compare the lengths of EM and MF but she does not drag the point P on the median]

28. Tutor: Ara varia P per veure que si la propietat es compleix per altres punts de la mitjana

29. Marta: Sí són iguals…

30. Tutor: Com ho podries demostrar?

In the following problems (median problem and quadrilateral problem), the students understand the logical structure of the problem (which are similar, i.e. the statement has a structure of two nested universal quantifiers). They start to consider varying the quantified point P and the quantified triangle.

Moreover, the students connect conceptual structures, as they connect Thales theorem with similarity of triangles. In the quadrilateral problem, Marta states ‘serien triangles semblants. Bé, és Thales això. Si sabem que això és la meitat d’això, això és la meitat d’això’ (Figure 5.1d). They also discern geometric properties of the figure from properties of the drawing. For instance, in the median problem Marta states: ‘és que aquesta línia [median] parteix el triangle per la meitat [two congruent triangles] però clar, això depèn del triangle…’ [She drags the vertices B and C to obtain a general triangle as a counterexample]. In the quadrilateral problem, Marta draws different
triangles to identify geometric properties (equality of heights in this case) and states: ‘Sí fas un triangle rectangle es veu. Clar és un exemple…’ She considers a different triangle and states ‘no, però mira això… [diferent heights (perceptual apprehension), Figure 5.1c] ..Clar, però aquí no és exacte…’

In the following table (Table 5.2) we report the different codes concerning the structural competence for each local cycle for the case of Marta.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning structural competence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(o) Partial understanding of the logical structure of the problem (static problem)</td>
</tr>
<tr>
<td></td>
<td>(+) Awareness of the properties of the diagonal and the properties of the area function as a measure function</td>
</tr>
<tr>
<td></td>
<td>(with the help of the tutor)</td>
</tr>
<tr>
<td>Scaled triangles problem</td>
<td>(o) Partial distinction between drawing and figure</td>
</tr>
<tr>
<td>(GGB)</td>
<td>(o) Intuitive identification of the centroid, but difficulties in identifying its geometric properties</td>
</tr>
<tr>
<td></td>
<td>(trisection of the medians)</td>
</tr>
<tr>
<td></td>
<td>(o) Applying Thales theorem in two-steps with the help of the tutor (she applied blindly Thales theorem)</td>
</tr>
<tr>
<td></td>
<td>(+) Understanding of the logical structure of the problem (two quantifiers) with the help of the tutor,</td>
</tr>
<tr>
<td></td>
<td>the task and the use of GGB</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Understanding of quantifier: ‘P any point of the median’, ‘any triangle’</td>
</tr>
<tr>
<td></td>
<td>(+) (Discursive apprehension) Congruence of alternate angles</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure (counterexample)</td>
</tr>
<tr>
<td></td>
<td>(o) Congruence criteria ASA (she does not consider the adjacent side S but she identifies in the figure</td>
</tr>
<tr>
<td></td>
<td>that both hypotenuses are congruent)</td>
</tr>
<tr>
<td>Quadrilateral problem</td>
<td>(+) Distinction between drawing and figure</td>
</tr>
<tr>
<td>(P&amp;P)</td>
<td>(+) Understanding of the quantifiers ‘any point P’, ‘any triangle’</td>
</tr>
<tr>
<td></td>
<td>(+) Relation between parallelism and proportional segments</td>
</tr>
<tr>
<td></td>
<td>(o) Relation between Thales theorem and similar triangles (usual configuration)</td>
</tr>
</tbody>
</table>

Table 5.2: Scores concerning the visualization competence (Marta)

**The instrumental competence**

Through the course of the teaching experiment, there is evidence of the students’ awareness of the necessity of proof, which may be exemplified by the following episodes:

In the scaled triangles problem, Marta applies Thales theorem trying to prove the equality of the segments (EM=MF). She states ‘Però com ho demostres si totes les lletres són diferents? Puc agafar nombres d’aquí per comprovar-ho.’

With the help of the tutor, she applies twice Thales theorem (transitivity of equality of ratios) and she states:

Marta: Per tant la relació entre EM i MF és que són iguals… \[
\frac{BM}{EM} = \frac{MC}{MF}
\] Però, podria ser que no fossin iguals i que fossin proporcionals…

Tutor: Què aporta el fet que M sigui punt mig del segment BC?
Marta: però això (BM=MC) perquè ho veig en el dibuix
Tutor: AM és la mitjana

Nevertheless, it should be pointed out that in the quadrilateral problem, some students have difficulties understanding the necessity of proving Thales reciprocal. Marta says ‘sí, sí, sempre ho és així’.
We also have evidence of a progressive search for conjectures and validation of conjectures. An example is the case of Marta. Whereas in the root problem there is an absolute lack of conjecturing (she tries to apply blindly Thales theorem in standard configurations), in the following problems she starts to pose and validate conjectures. Evidences of improvement are:

- Conjecturing with the help of GGB (in the scaled triangles problem she conjectures that $P$ is the centroid of the triangle).
- Elaboration of key-lemmas: in the median problem, she solves the problem modulo the equality of heights (as we will see later, GGB and tutor’s orchestration reveal to be important in this improvement)
- Use of strategies of proof-search: for instance, in the scaled triangles problem, Marta applies the heuristic ‘let’s suppose the problem solved’.

As we mentioned before, the tutor orchestration and the use of GGB play a key role in the experiment in the case of non-confident profile. An example is Marta’s resolution of the median problem. At first, she considers the inner heights of the triangles and she has difficulties in finding a resolution strategy. The use of GGB (information provided by the software, use of the dragging tool, etc.) and the tutor’s messages suggest to Marta the elaboration of a key-lemma.

Nevertheless, we notice that the use of GGB is limited, for the students only drag elements partially in a non-confident way. Despite the ‘user friendly character’ of GGB, the students from this profile have a poor knowledge of some fundamental features of this software, namely, the interpretation of the draggability of objects and the problem of the round-off error awareness (GGB plane as a discrete non-continuous plane).

In the following table (Table 5.3) we report the different codes concerning the instrumental competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning instrumental competence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(-) She does not consider stating conjectures about the relation of areas</td>
</tr>
</tbody>
</table>
| Scaled triangles problem (GGB) | (o) Understands the necessity of proof despite proposing a naïve justification (lines 36-38)  
(+ ) She tries to state a conjecture about the position of the point P (lines 46-51) |
| Median problem (GGB)           | (o) Construction of an isosceles triangle as general case (but proof not dependent on the drawing) (lines 22-30) 
(+ ) Understands the necessity of proof (lines 4-7)  
(+ ) Search for a counterexample (lines 27-30) |
| Quadrilateral problem (P&P)    | (o) Counterexample (based on perceptual properties of the drawing) (lines 28-30, Figure 4.17)  
(+ ) Conjecture (equality of areas) with the help of the tutor (lines 47-48)  
(o) Conjecture (which is only verified for a concrete triangle) (Figure 4.15, line 5) |

Table 5.3: Scores concerning the instrumental competence (Marta)

**The deductive competence**

There is no evidence of a shift in the deductive competences. In the root problem, the students are able to prove the equality of areas using a strategy based on applying equicomplementary dissection rules, but they are not successful transferring the verbal reasoning to a written deductive valid text. In the other problems they have also
difficulties in elaborating a deductive justification. Despite the fact that they show certain local abilities for deductive steps, they are not capable of relating deductive steps into what would be a whole deductive reasoning.

For instance, in the scaled triangles problem, Marta proposes a naïve justification to justify that the centroid is the point that verifies the required property. She has difficulties visualizing the trisection of a segment in a non-standard position of the segments. Nevertheless, she is aware of the fact that these naïve justifications are not accepted as proofs. For instance, she states in the second problem: “*Com ho demostres si totes les lletres són diferents? Ho podria comprovar amb nombres d’aquí*” [use of measure tools].

In the following table (Table 5.4) we report the different codes concerning the deductive competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning deductive competence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(+) She produces a pragmatic justification (generic example) but she is not able to write it down in the worksheet (lines 29-33)</td>
</tr>
</tbody>
</table>
| Scaled triangles problem (GGB) | (+) Deductive justification for the first question  
(-) She stays at a spatial-graphical field (written protocols). Naïve justification |
| Median problem (GGB)     | (+) Attempts a sequence of deductive steps (correct reasoning. She uses the criteria SAS as tool of her deductive reasoning but the criteria is applied wrongly).                       |
| Quadrilateral problem (P&P) | (+) Deductive reasoning (Figure 4.20: worksheet)                                                                                                                          |

| Table 5.4: Scores concerning the deductive competence |

5.1.1.2 Results on the HLT: confident profile

These students are high achieving students who are more confident about their ability to solve the problem proposed than with the previous profile. They have an ‘analytic’ profile (Krutekskii, 1976). In this section, we formulate the feed-forward of the hypothetical learning trajectory.

During the teaching experiment, there are no relevant changes in these students’ cognitive structures concerning the procedures used to solve the problems. They do not use strategies based on discerning equivalent triangles and applying properties of measure as a measure function. They discern congruent, similar triangles and equivalent triangles, but they do not consider strategies based on applying equicomplementary dissection rules. They tend to label elements of figures to state algebraic relationships between these elements. They apply the formula of the area of the triangle to obtain the required relation between the areas as a ratio of expressions.

The visualization competence

There is a shift in the visualization competence. In the root problem, these students discern similar and congruent triangles, but they do not visualize dynamic variations of the point E on the diagonal. At first they try to obtain the concrete values of the rectangles’ areas. With the help of the tutor the students conjecture the equality of heights but they do not consider other positions of the point E on the diagonal, as we can see in the paragraph below:

13. Guillem: Et queda lo mateix! [He gets only one equation when he tries to solve the system of two linear equations and two unknowns: the system has infinite solutions (all the points of the diagonal)]
There is a shift in the visualization competence, as these students start to visualize dynamic variations of the elements (mental reconfigurative operative apprehension). In the quadrilateral problem, Guillem visualizes the variation of the point P along the base of the triangle, as we can see in the paragraph below:

6. Guillem: Si sé que l’àrea de la suma dels dos no varia vol dir que l’altra [BNMP] no varia per molt que tingui el punt P. L’únic canvi és que un perd base [AMP] i l’altra en guanya [NPC] però les altures en principi han de ser fixes

Nevertheless, immediate perceptual approach can be an obstacle for these students. For instance, in the median problem Guillem conjectures through immediate perceptual approach a false conjecture and does not try to validate it before starting an execution phase trying to prove deductively the conjecture. Also these students start to use figural inferences and to base their reasoning on figures (see the paragraph above). Nevertheless, they still have visualization difficulties, such as: discerning Thales configuration in non-standard positions, visual-algebraic difficulties. In the scaled triangles problem, they have visual-algebraic difficulties and in the quadrilateral problem they do not visualize the relation between the area of the inside quadrilateral and the area of the outside triangle. This is again a visual-algebraic difficulty.

In the following table (Table 5.5) we report the different codes concerning the visualization competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning visualization competence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(+) Operative apprehension: he extracts three similar right-angled triangles</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing equivalent figures and equicomplementary dissection rules</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(-) Difficulty to visualize the relation between the trisection of the segment BM with the trisection of the base BC which is twice the segment BM (resolution process)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing the ratio 1:3 in the segment AM (median)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension: he introduces another median and parallel lines</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Dynamic Visualization (varying P along the base)</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing the exterior heights of a triangle</td>
</tr>
<tr>
<td></td>
<td>(-) Immediate perceptual approach of the triangles as an obstacle for the geometric interpretation (wrong conjecture)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (he extracts two triangles from the initial configuration)</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) He draws a ‘general’ triangle in an oblique position</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (dynamic visualization)</td>
</tr>
<tr>
<td></td>
<td>(o) Difficulty in visualizing Thales configuration</td>
</tr>
<tr>
<td></td>
<td>(-) Difficulty in visualizing the relation between the areas of the inner figures and the outside triangle</td>
</tr>
</tbody>
</table>

Table 5.5: Scores concerning the visualization competence
The structural competence
There is a relevant shift in the structural competence. These students had difficulties in the root problem and they progressively connect conceptual structures. The tutor’s orchestration, the similar tasks and the synergy of environments foster adequate insight into the structural competence. The students progressively understand the logical structure of the problems and connect conceptual structures concerning Thales theorem and similar triangles. They discern congruent, similar and equivalent triangles and overcome obstacles such as the concept of exterior height of a triangle (see Guillem’s resolution of the median problem in the fourth chapter). Despite the fact that they discern equivalent triangles, they do not connect different resolution approaches based on equicomplementary dissection rules. There are no relevant changes in these students’ cognitive structures concerning the procedures used to solve the problems. They focus on applying the area formula of triangles and obtaining the expressions of the areas to compare. Nevertheless, they understand the parameter role in these expressions and they do not try anymore to obtain concrete values. This is due to the understanding of the logical structure of the problems (quantifiers). For instance, in the root problem Guillem tried to obtain the concrete lengths of the sides of the rectangle, as we can see in the paragraph below:

Guillem: podria utilitzar proporcionalitat, una regla de tres ...però no coneix això.. [AE]... [Guillem stares at the figure in silence for a while]
Guillem: Jo agafaria aquest triangles petit... [AME, Figure 5.3]
Guillem: [He writes the ratio between the sides of three similar triangles (Figure 5.3)]. Ho podria resoldre amb un sistema d'equacions

\[
\frac{c}{y} = \frac{x}{5} \Rightarrow \frac{c}{5} = \frac{5x}{x-y} \Rightarrow c(5) = \frac{5c}{x-y} \\
\frac{z}{y} = \frac{5-x}{x-y} \Rightarrow z = \frac{5}{x-y} \Rightarrow g \times z = g \times -z
\]

Figure 5.3: root problem

In the median problem, which has a verbal statement and no references to concrete values, Guillem states ‘Com que no fa referència a dades concretes puc agafar un triangle rectangle’. Guillem reacts to the tutor message by considering an isosceles triangle. Nevertheless, his reasoning is not dependent on the particular properties of the isosceles triangle. This reveals a lack of understanding concerning the universal
quantifier. Instead of considering a universal quantifier, he considers an existential quantifier.

Instead, in the quadrilateral problem (Figure 5.4) Guillem considers a generic triangle (the usual way of proving a universal quantifier statement). He also understands that he can not obtain the concrete lengths of the elements of the triangles, as we can observe in the following paragraph. Nevertheless, Guillem does not solve his initial difficulties concerning the system of equations (root problem); he does not build the meaning.

Guillem: Jo crec... que sempre tenen la mateixa àrea perquè aquí les altures no varien...Saps que vull dir. els dos triangles...Bueno, no sé... [areas of the triangles AMP and PNC (Figure 5.4)]

[Guillem observes the figure in silence]

Guillem: Si sé que l’àrea de la suma dels dos no varia vol dir que l’altre [BNMP] no varia per molt que tingui el punt P. L’únic canvi és que un perd base [AMP] i l’altre en guanya [NPC] però les altures en principi han de ser fixes

The problems proposed in a more generic way, independent of concrete lengths, the tutor’s orchestration and the synergy of environments all foster the understanding of the logical structure of the problems.

In the following table (Table 5.6) we report the different codes concerning the structural competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning structural competence (second case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(-) Lack of understanding of the logical structure of the problem</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(+) Understanding of quantifier: ‘P any point of the median’</td>
</tr>
<tr>
<td></td>
<td>(o) Thales property applied without a specific strategy</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Conception of the height as perpendicular line to the base of the triangle (instead of the line that contains the base).</td>
</tr>
<tr>
<td></td>
<td>(+) Understanding of quantifier: ‘P any point of the median’</td>
</tr>
<tr>
<td></td>
<td>(+) Search for conjectures and partial validation: Conjectures $C_1$, $C_2$, $C_3$</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension [Trigonometry of the right-angled triangle to justify congruence of two inside triangles, properties of the area]</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) Distinction between drawing and figure</td>
</tr>
<tr>
<td></td>
<td>(++) Understanding the logical structure of the problem (any triangle, any point)</td>
</tr>
<tr>
<td></td>
<td>(+) Search for conjectures and effort to validate them</td>
</tr>
<tr>
<td></td>
<td>(+) Discursive apprehension: Equivalence parallel lines and angle congruence, Thales theorem, properties of the area function.</td>
</tr>
<tr>
<td></td>
<td>(+) Conjecture (equal areas)</td>
</tr>
</tbody>
</table>

Table 5.6: Scores concerning the structural competence

The instrumental competence

There is also a shift in the instrumental competence. There is evidence of the students’ awareness of the necessity of proof in the root problem and during the teaching experiment. The students try to apply well-known properties and theorems to justify deductively the steps followed. We also have evidence of an improvement of the proof-search strategies. Evidences of improvement are:

- Conjecture with the help of GGB: In the root problem, Guillem does not try to state any conjecture and follows an algebraic strategy trying to solve a linear system of two equations and two unknowns. He is not able to interpret the
solution obtained, as he is trying to obtain the concrete values of the areas. Instead, in the following problems he poses and validates conjectures.

- Progressive use of elaborated figural inferences. For instance, in the quadrilateral problem we have evidence of the following figural inferences:

\[ \text{‘quan un perd base l’altre en guanya I (Figure 5.4) } \]
\[ \text{les alçades són igual (perceptual apprehension) } \]
\[ \text{→ (AMP)+(PNC) = constant} \]

Guillem applies also a figural inference to obtain the relation between the segments MN and AC (Figure 5.4), as we can see in the paragraph below:

41. Guillem: Jo crec que hauria de dir, bé imagino, que MN és un mig, és la meitat de AC.
42. Tutor: Què vols dir?
43. Guillem: Clar perquè, aquí l’àrea seria x per MN i llavors multipliques per 2 [Figure 6].

Perquè el quadrilàter tingui la mateixa àrea que els dos triangles… [Equal areas → MN = \( \frac{1}{2} \) AC]

- Elaboration of key-lemmas: Guillem solves the median problem modulo the equality of heights.

The use of GeoGebra and the tutor’s orchestration play a key role in this improvement. The tutor’s messages encourage Guillem to pose conjectures. As an example, we show the following paragraph (resolution of the root problem):

13. Guillem: Et queda lo mateix! [he obtains the same equation from the initial system of two linear equations]
14. [He stares at the figure in silence. He is lost]
15. Tutor: No calen dades concretes per resoldre aquest problema
16. Guillem: No sé... Em sembla que tenen la mateixa àrea
17. Tutor: Perquè creus que tenen la mateixa àrea?
18. Guillem: No sé,… perquè lo que perd un de un de llargada o guanya d’amplada

The use of GGB also helps the students to pose and test conjectures. Nevertheless, immediate perceptual approach can be an obstacle for the students. In the median problem, Guillem tries to prove the false conjecture and does not try to validate the conjecture. The tutor’s message helps Guillem to refute the conjecture. The students progressively understand the need of validating the conjectures. For instance, in the quadrilateral problem Guillem conjectures that the line through the midpoints is parallel to the base of the triangle and tries to validate the conjecture considering several triangles, as we can see in the paragraph below:

8. Guillem: Jo crec que l’altura sempre és... Es poden unir els punts mitjós amb una recta paral·lela a aquesta [AC] i això em diria que l’altura és igual...
9. Guillem: …Però, clar aquí sense el GeoGebra no ho saps… a lo millor el punt mig és una mica més avall…
10. Tutor: Què vols dir amb això?
11. Guillem: Tu pots imaginar que una paral·lela pugui passar per M i N però no saps segur si el punt mig està a la paral·lela o una mica més avall. Com que no treballem amb ordinador…
We could interpret the above paragraph as an evidence of the fact that the use of dynamic geometry software obstructs the necessity of proving, because the student would be convinced about the truth of the parallelism when working with GGB. Nevertheless, we believe that this is due to the fact that Guillem has in fact difficulties in understanding the proof of Thales reciprocal. We should not then to infer that the use of DGS obstructs the necessity of proof.

In the following table (Table 5.7) we report the different codes concerning the visualization competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning instrumental competence (second case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(o) Figural inference based on analogy (the proposition intended by the figural inference is false).</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(+) Understands the necessity of proof</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(o) Construction of an isosceles triangle as general case, he does not validate the geometric properties in other triangles (but proof not dependent on the drawing)</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) Figural inference (elaborated figural inference whose intended proposition is true)</td>
</tr>
</tbody>
</table>

Table 5.7: Scores concerning the instrumental competence

The deductive competence
We have no evidence of a relevant shift in the deductive competence. Nevertheless, these students already have insight into this competence as a starting level in the hypothetical learning trajectory. These students have difficulties in elaborating deductive justifications. At first, they try to follow algebraic resolutions, but they have difficulties in interpreting the results obtained. In the root problem, they have difficulties in identifying the tautology obtained.

During the teaching experiment they mainly produce crucial experiments. They make an explicit use of the logical form ‘if …then’ in the formulation and handling of conjectures, as well as using logical rules.

In the quadrilateral problem, they have difficulties to prove Thales reciprocal. This is due to a lack of awareness about the formal strategy of proving by contradiction.

Nevertheless, there is a progressive understanding of the necessity of considering generic examples, which is at the root of a formal proof involving a universal quantifier. For instance, in the median problem Guillem states: ‘Com no diu res puc considerar un triangle rectangle’. Instead, in the quadrilateral problem, Guillem tries to consider a generic triangle. He draws a general triangle with an oblique position on the worksheet.

In the following table (Table 5.8) we report the different codes concerning the visualization competence for each local cycle.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning deductive competence (second case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem</td>
<td>(o) Deductive steps to prove the equality of areas</td>
</tr>
</tbody>
</table>

Table 5.8: Scores concerning the deductive competence
Table 5.8: Scores concerning the deductive competence

<table>
<thead>
<tr>
<th>(P&amp;P)</th>
<th>(+) Use deductive reasoning by logical inferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scaled triangles</td>
<td>(o) Reasoning based on the properties of the centroid, validated empirically (non proved key-lemma)</td>
</tr>
<tr>
<td>problem (GGB)</td>
<td>(+) Deductive reasoning to obtain a key-lemma</td>
</tr>
<tr>
<td></td>
<td>(+) Deductive reasoning to elaborate a proof</td>
</tr>
<tr>
<td>Median problem</td>
<td>(o) Difficulties in proving Thales reciprocal: case-based reasoning</td>
</tr>
<tr>
<td>(GGB)</td>
<td>(+) Deductive reasoning</td>
</tr>
<tr>
<td>Quadrilateral problem</td>
<td>(P&amp;P)</td>
</tr>
<tr>
<td></td>
<td>(o) Difficulties in proving Thales reciprocal: case-based reasoning</td>
</tr>
<tr>
<td></td>
<td>(+) Deductive reasoning</td>
</tr>
</tbody>
</table>

5.1.1.3 Results on the HLT: autonomous profile

These students are high achieving students who are autonomous. They tend to think in terms of pictures and we may consider that they have a ‘geometric’ profile (Kruteskii, 1976). These students are confident on their ability to solve the problems and to use the software. During the teaching experiment, there are changes in these students’ cognitive structures concerning the procedures used to solve the problems. Despite the fact that they tend to base their resolution strategies on visual thinking, they connect different resolution approaches. They try to discern congruent, similar and equivalent triangles and are aware of the properties of the area function as a measure function. As a last resort, in the resolution process of a problem, they have the confidence to apply analytic geometry and algebraic approaches based on considering equality of ratios and applying the area formula of a triangle.

The visualization competence

These students prefer visual-based thinking, and despite the fact that they have a high degree of competence as a starting level, there is a shift in their visualization competence concerning dynamic visualizations, in the particular context of the problems proposed. In the root problem, they apply directly equicomplementary dissection rules and do not pay attention to the concrete lengths of the outside rectangle. For the second problem, we notice a shift concerning the (+) scores. The scaled triangle problem and the use of GeoGebra foster the visualization of algebraic-geometric properties of segments as quantities. For instance, Aleix visualizes the relation between the segments defined by the point P (centroid) on the median in the scaled triangles problem, as we can see in the paragraph below.

Aleix: vull veure que és un terç…però no pot ser surt massa petit [segment MP]
[Aleix observes now the equation of the circumference on the algebraic window. He points at the centre coordinates and at the squared radius.]
[…]
[Aleix constructs the circumference of center A and radius d/3 (d is the length of AM)]
Aleix: tampoc sembla ser el resultat! És massa petit…
[Aleix drags the vertices of the triangle and refutes the previous construction]
Aleix: Ah! Vale! Primer..per a que la circumferència... per a que l’espai que no avarca sigui un terç, he de posar dos terços! [Figure 5.5]
Nevertheless, in the scaled triangles problem, they do not visualize the algebraic relationships between the trisection of the segment BC and the trisection of the segment BM. For instance, in the verification phase Aleix tries to consider other resolution strategies. He has considered a resolution approach based on discerning similar triangles. He states ‘També, he pensat fer-ho amb Thales, però no veig com. S’hauria de fer en dos passos...em sembla’.

Concerning the transition with the following micro-cycle (problem of the median), the (+) scores do not reveal a shift in the visualization competence. This is due to the fact that in the previous problems, the students had to discern congruent or similar figures, whereas in the third problem they have to discern equivalent triangles. In this case, the mediation of the drag tool combined with measure tools (area measure) fosters the dynamic visualizations. The students drag the elements of the figure in a wide area, and this leads to the key idea of applying a resolution strategy based on equicomplementary dissection rules. The results show the influence of the third problem and the use of the drag tool on the visualization processes observed in the quadrilateral problem (reconfigurative operative apprehension). For instance the students apply visual transformations in terms of areas, as we can see in the paragraph below.

Aleix: Podem agafar aquests triangles [AMP i PNC] i dir que és un sol triangle imaginari de base AC i multiplies la base AC per l’alçada i divideixes per dos. [Figure 5.6]
Tutor: que vols dir amb triangle imaginari?
Aleix: bé, tinc dos espais [both triangles] i dic que només ho considero com un sol espai.
The students also visualize invariants such as constant areas. This leads to the consideration of resolution strategies based on particularization and further generalization based on dynamic strategies.

Concerning this local cycle, we identify transitions to a higher degree of competence. The proposed tasks and the mediation of the software facilitate these transitions. With the following tables we have an overview of the global learning trajectory concerning the visualization competence.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning visualization competence (third case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(+) Operative apprehension (extraction of congruent triangles)</td>
</tr>
<tr>
<td></td>
<td>(+) Operative apprehension (extraction of similar rectangles)</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(+) Operative apprehension (extraction of similar triangles and visualization of homothetic transformation)</td>
</tr>
<tr>
<td></td>
<td>(+) Partial visualization algebraic-geometric (segments defined on the median by the centroid)</td>
</tr>
<tr>
<td></td>
<td>(+) Distinction between drawing and figure</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Operative apprehension (extraction of equivalent inner triangles)</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) Dynamic visualization: the area of MNP does not depend on P (invariants)</td>
</tr>
<tr>
<td></td>
<td>(+++) Reconfigurative operative apprehension (transformation of two shapes in an equivalent shape)</td>
</tr>
</tbody>
</table>

Table 5.9: Scores concerning the visual competence

**The structural competence**

These students have highly connected conceptual structures and a high degree of competence as a starting level of the hypothetical learning trajectory. They understand the essential elements of the problems and the logical structure of the problems. These students discern superfluous numerical data of the statement of the problem. In these problems, numerical data inserted in the verbal statement or in the figure of the statement are not relevant. For instance, in the scaled triangles problem the figure of the statement is inserted into a grid to help the students to overcome difficulties related to visual-algebraic relations between segments as quantities. In this problem, the simultaneous use of the algebraic and geometric window helps Aleix to understand the relation between segments as quantities. Through the instrumented scheme ‘dragging to find geometric-algebraic invariants’, Aleix deduces the ratio between the segments on the median.

The instrumental genesis of the software helps these students to understand in depth the dependence relations between the elements of the problems. For instance, we observe the instrumented scheme ‘algebraic-geometric invariant guessing’. They use this scheme to understand the motion dependency of the elements of the figure. As an example, in the scaled triangles problem the instrumentation of the drag tool helps Aleix to understand homothetic transformations. He is near to developing awareness of the homothetic transformation between the triangles EPF and ABC. The root problem and the tutor’s message also help Aleix to grasp the homothetic relation, as we show in the following paragraphs.

In the root problem, Aleix states that the rectangles ABCD, EPCO and ENAM (Figure 5.7) are similar, as we can see in the paragraph below:

6. Aleix: EPCO [Figure 5.7] és un rectangle similar al gran perquè té tots els costats paral·lels
7.1. Tutor: Estàs segur?
   […]

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7.2. Tutor: és sufficient que els costats siguin paral·lels per tal que les figures siguin similars?
8. Aleix: […] Si els angles són iguals, la forma ha de ser la mateixa
10. Aleix: però, vale… [He remains in silence looking at the figure]
11. Aleix: vale… Com aquí el cas és diferent…
12. Tutor: A què et refereixes?
13. Aleix: la diferència és aquesta creu [lines MO and NP that intersect at E], E baixa per la diagonal AC… Aquesta creu és el que fa que siguin proporcionals. Tenen la intersecció en el mateix punt [Figure 5.7]

Figure 5.7: labelled figure associated to the statement of the root problem

Figure 5.8: Aleix tries to construct the similar triangle (EPF) by applying the tool dilatation

Aleix reacts to the tutor’s message (line 9) by considering the transformation between the rectangles ABCD and EOCP (line 13). In the scaled triangles problem, Aleix identifies again similar polygons and the transformation between them, as we can see in the paragraph below. Moreover, he tries to apply twice the tool dilation (Figure 5.8), but finally he decides to construct the inner triangle by applying the tool circle given the centre and the radius (to obtain the trisection of the median). We interpret that Aleix is near to developing awareness of the homothetic transformation between the triangles, but he has not appropriated the instrumented scheme.

12. Aleix: vale, té tres costats paral·lels per tant són proporcionals. En un triangle si que val […]
17. Aleix: vale, a veure, com són proporcionals, hem de fer tal proporció que la base de EPF sigui una tercera part de la base de BC…
18. Aleix: Ja que són triangles proporcionals, podem disminuir un costat. Per exemple l’alçada… [he points at AM]
21. Aleix: per tal que un costat sigui la tercera part de BC jo he de fer que AP sigui una tercera part de…
22. [Aleix tries to trisect the median with a robust GGB construction]
23. [Aleix searches in the toolbar the different tools and observes all the tools in each icon. He returns to the tool dilatation twice but he abandons it] (Figure 5.8)

The local cycle (third itinerary of problems) also facilitates a deeper understanding of the properties of area function as a measure function (see the resolution of the median problem in the fourth chapter) and the similarity properties.

In the following table we have an overview of the global learning trajectory concerning the structural competence.
<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning structural competence (third case)</th>
</tr>
</thead>
</table>
| Root problem (P&P) | (+++) Deep understanding of the logical structure of the problem  
(+ ) Analogy criteria AAA for rectangles (Deeper understanding of the criteria)  
(+ ) Discursive apprehension: property of the diagonal (it splits a triangle into two equivalent triangles), properties of the area function as measure function |
| Scaled triangles problem (GGB) | (+++) Deep understanding of the logical structure of the problem  
(+) Discursive apprehension: similarity criteria for triangles, similarity ratio  
(+) Distinction between drawing and figure (exploration of round-off error behaviour by dragging to validate a geometric property) |
| Median problem (GGB) | (o) False Conjecture through perceptual approach (congruence of triangles) based on a particular case (conviction of the truth of the conjecture without validating in other triangles)  
(++ ) Understanding of the motion dependency in terms of logical dependency and the logical structure of the problem  
(+) Discursive apprehension: Criteria ASA, property of the median (it splits a triangle into two equivalent triangles), properties of the area function as measure function (resolution process) |
| Quadrilateral problem (P&P) | (+++) Deep understanding of the logical structure of the problem  
(+) Discursive apprehension: Thales theorem, Thales reciprocal, definitions, properties of the area function as measure function |

Table 5.10: Scores concerning the structural competence

**The instrumental competence**
These students internalise the dragging as a theoretical control during the teaching experiment. They develop instrumented schemes such as ‘algebraic-geometric invariant guessing’. The use of a technological environment (GeoGebra and paper-and-pencil) fosters the search for elaborated conjectures.

For instance, the instrumentation of the dragging tool allows Aleix to search for algebraic-geometric invariants in the scaled triangles problem.

The instrumental competence is boosted by the software. An evidence of this claim is the massive dragging, which Aleix uses in the scaled triangles problem (he tries to observe the round-off behaviour to identify a false negative situation) and in the median problem (he changes the orientation of the figure).

In the median problem, the use of GGB helps Aleix to find a resolution strategy based on equicomplementary dissection rules. At first, the student tries to follow a trigonometric strategy, but the student reacts to the tutor’s message by applying massive dragging in order to identify geometric properties, as we can see in the paragraph and the figure below. He changes the orientation of the triangle, as we can see in the figure below.

21. Aleix : És possible resoldre aquest problema amb trigonometria?  
22. Tutor: tracta de buscar invariants amb l’ajuda de GeoGebra  
23. Aleix: si les àrees i les bases són iguals, les alçades han de ser iguals.  
24. Tutor: Intenta descompondre el triangle en altres triangles  
25. [Aleix does not react to this message and tries to find again a trigonometric resolution. He is lost]  
[Aleix abandons the paper-and-pencil strategy and initiates the massive dragging in order to search for a solving strategy (Figure 5.9)]
In the following table we have an overview of the global learning trajectory concerning the instrumental competence.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning instrumental competence (third case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(+) Conjecturing</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(+++) Spontaneous conjecturing and self-initiated effort to deductively verify the conjectures</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Understanding of the necessity of proof</td>
</tr>
<tr>
<td></td>
<td>(+) Search for counterexamples (dragging the elements of the triangle ABC to obtain different cases of triangles APC and APB)</td>
</tr>
<tr>
<td></td>
<td>(+) Key-lemma equivalent to the problem</td>
</tr>
<tr>
<td></td>
<td>(+++) Search for geometric invariants by dragging in a large area all the elements (Cinema dragging)</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) Search for conjectures</td>
</tr>
<tr>
<td></td>
<td>(+) Search for conjecture-lemma</td>
</tr>
</tbody>
</table>

**Table 5.11:** Scores concerning the instrumental competence

**The deductive competence**

These students have special ability reasoning. They are able to produce deductive justifications, as we can see in the problem’s resolutions. Nevertheless, there is a shift in the deductive competence concerning the awareness of different formal strategies of proving. For instance, in the quadrilateral problem they partially understand the proof of Thales reciprocal with the help of the tutor. They develop partial awareness of the proof by contradiction.
10. Aleix: en un triangle els punts mitjos estan en la paral·lela a la base AC [Figure 5.11]
11. Tutor: Ho sabries justificar?
12. Aleix: L’alçada és la mateixa, com que sé que l’alçada és perpendicular i la longitud és la mateixa, faig la recta per dos punts i és paral·lela…o sinó amb Thales.
13. Aleix: si les rectes són paral·leles i ...
14. Aleix: Com els segments són iguals (midpoints) podem dir que les rectes són paral·leles per Thales.
15. Tutor: Que passaria si les rectes no fossin paral·leles?
16. Aleix: No serian segments iguals, suposo…

In the problem’s worksheet Aleix writes the following chain of implications, which shows, as we have already mentioned, a partial awareness of the proof by contradiction (Figure 5.10).

In the following table we have an overview of the global learning trajectory concerning the deductive competence.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Coding scores concerning deductive competence (third case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root problem (P&amp;P)</td>
<td>(+) Deductive reasoning to prove the conjecture.</td>
</tr>
<tr>
<td>Scaled triangles problem (GGB)</td>
<td>(+) Deductive reasoning to prove the conjecture.</td>
</tr>
<tr>
<td>Median problem (GGB)</td>
<td>(+) Use of key-lemmas&lt;br&gt; (+++) Deduction rigorously formulated with paper and pencil</td>
</tr>
<tr>
<td>Quadrilateral problem (P&amp;P)</td>
<td>(+) Awareness of the distinction between implication and double equivalence, transitivity of the implication, proof by contradiction&lt;br&gt; (+++) Proof by equivalence</td>
</tr>
</tbody>
</table>

Table 5.12: Scores concerning the deductive competence

5.1.2 Results on the orchestration

In this section we formulate the results on the orchestration for each prototypic case (not confident, confident, and autonomous students).

5.1.2.1 Results on the orchestration: ‘Not confident’ students

These students are not confident in their ability to solve problems and frequently ask for validation messages. For instance, in the root problem Marta has found results that can lead her to a resolution strategy, but she is not certain about her strategy. She is afraid to analyse her initial ideas in depth. In this problem she states, ‘És que, és molt difícil! És que potser no vaig bé… Els altres han resolt el problema? Vaig bé?’

The tutor tries to elicit students thinking without giving relevant information. They frequently ask for messages during the familiarization phase and often start local
evaluation phases during the resolution process. We identify two different situations in the orchestration: the students request the help of the tutor and the tutor gives a non-requested message.

As mentioned before, these students often require cognitive messages in the familiarization phase. They have difficulties in understanding the logical structure of the problems. The tutor’s orchestration has a key role to foster this understanding. They also request often contextual messages because they are not confident using the software. For instance, in the second problem, which is of a constructive nature, Marta is aware of the necessity of making robust constructions, but she has difficulties in trisecting the segment with GeoGebra. She states ‘Però, no es pot fer a ull...això no queda exacte.... No es pot veure aquí?’ [algebraic window]. The tutor’s orchestration is relevant for the instrumental genesis of the software. Nevertheless, the tutor’s messages during the didactical performance tend to be messages of level two and three. It would have been better if the tutor had given only messages of level one and level two to elicit students’ exploration of the software. For instance, in the resolution of the median problem, Marta has difficulties constructing the feet of the exterior height of a triangle. She uses the tool intersection of objects applied to a segment (the base of the triangle) and to a perpendicular line. The software allows this construction and thus Marta does not obtain an error message. She does not observe that on the algebraic window there is the constructed object, which is an *undefined point*. This technical difficulty is related to Marta’s conception of the height as the perpendicular line to the base (instead of the line that contains the base). The tutor gives Marta a contextual message of level three ‘Has de considerar la recta per obtenir el punt d’intersecció’. It would have been better if the tutor had given first a message of level one. These students tend to react to messages of level one by asking for more information. The interactions with the tutor are usually dialogues with more than four interventions. For instance, in the resolution of the quadrilateral problem Marta states the conjecture to the tutor in order to validate it. The tutor gives Marta a cognitive message of level one and Marta reacts by asking again for validation messages. This may be due to a lack of confidence. As we see in the paragraph below, she has conjectured that both areas are equal.

\[
\begin{align*}
\text{Marta:} & \quad \text{la resta entre una àrea [MPNB] i l’altre [triangles AMP and PNC] sempre serà constant.} \\
\text{Els altres dos [both triangles AMP and PNC] ja he vist que és constant} \\
\text{Tutor:} & \quad \text{Però quina relació tenen?} \\
\text{Marta:} & \quad \text{Que la resta és constant. Això no és una relació?} \\
\text{Tutor:} & \quad \text{Sí, però quina constant?} \\
[\ldots] \\
\text{Tutor:} & \quad \text{seria possible que el quadrilàter tingui la mateixa àrea. que la suma de les àrees dels 2 triangles?} \\
\text{Marta:} & \quad \text{És que jo diria que són iguals}
\end{align*}
\]

The second situation corresponds to the case in which the tutor gives a non-requested message. This happens if the tutor detects a lack of understanding or the use of an inadequate property, or if there is no feedback from the student. Nevertheless, this last situation is not usual, as these students tend to request the help of the tutor.
The tutor frequently gives messages because these students tend to abandon their initial ideas due to non-relevant difficulties. For instance, in the resolution of the scaled triangles problem, Marta is misled by the use of different labels in each environment. The tutor suggests that Marta rename the GeoGebra objects (contextual message of level one). Also, the tutor tries to give contextual messages to foster the use of the drag tool. During the teaching experiment, there is a shift concerning the use of the drag tool to validate conjectures or to find counterexamples and the tutor’s orchestration has a key role in the instrumental genesis of the drag tool.

5.1.2.2 Results on the orchestration: ‘confident’ students
These students are more confident about their resolution strategies and try to develop their initial ideas, even if they are not sure about them. For instance, in the root problem, Guillem tries to obtain the concrete lengths of the sides of the rectangles by solving a linear system of equations. He states ‘no estic segur, però per provar...’

The tutor lets the students follow their strategies without giving messages to elicit students’ thinking. Nevertheless, if they lose track of the solving strategy and may spend a great amount of time due to irrelevant mistakes, the tutor gives cognitive messages of increasing level.

These students usually react to messages of level one and level two, but they do not tend to ask for cognitive messages. They ask for validation messages in the familiarization phase, and when they are lost they ask for cognitive messages. For instance, in the quadrilateral problem, Guillem tries to justify that the triangles have the same heights, but he has difficulties with Thales reciprocal and requests a message from the tutor, as we can see in the paragraph below.

Guillem: Jo crec que l’altura sempre és... Es poden unir els punts mitjos amb una recta paral·lela a aquesta [AC] i això em diria que l’altura és igual...
Guillem: …Però, clar aquí sense el GeoGebra no ho saps...a lo millor el punt mig és una mica més avall....
Tutor: Què vols dir amb això?
Guillem: Tu pots imaginar que una paral·lela pugui passar per M i N però no saps segur si el punt mig està a la paral·lela o una mica més avall. Com que no treballo amb ordinador...
[Guillem draws many triangles to check the property: the line through the midpoint M, parallel to BC joins the midpoint N (Figure 5.12)]

Figure 5.12: Validation of Thales reciprocal in different cases

[...]
Guillem: Puc posar això? Com que sempre que faig una paral·lela de AC, passa per M i N vol dir que l’ altura sempre serà la mateixa pels dos triangles
For the problems to be solved with paper-and-pencil, the tutor gives mainly non-requested messages. Instead, for the problems to be solved in a technological environment, the students request more frequently contextual messages, but they also try to solve the problems on their own without the help of the tutor.

5.1.2.3 Results on the orchestration: ‘autonomous’ students
These students are used to working on their own and they are autonomous students. The tutor acts as a companion who observes the student resolution process to obtain feedback from the student’s actions. In this case, we may consider that these students work in parallel (Cobo, 1998). As part of the didactic performance, the tutor gives mainly messages of level one to make the students explain the steps in their resolution processes, and to elicit students thinking rather than giving them the solution to the difficulties encountered. We distinguish two different situations: firstly, the case in which the student requests the help of the tutor and, secondly the case in which the tutor gives a non-requested message. These students do not request the help of the tutor frequently. They do not tend to request cognitive messages because they want to solve the problem on their own. They rarely request the help of the tutor, and this occurs mainly in the familiarization phase to check that they have correctly interpreted the verbal statement of the problem. We observe this situation for problems whose statement includes a figure inserted in a grid. They do not pay attention to the grid and ask for validation messages. For instance, in the scaled triangles problem, Aleix understands the logical structure of the problem and asks the tutor: ‘E i F són punts qualssevol?’ In this case, the grid obstructs the understanding of the logical structure of the problem. Aleix does not pay attention to the grid and constructs a generic triangle with GeoGebra by clicking on three points, which have not integer coordinates, of the geometric window.

They also ask for contextual messages of level one related to usage schemes that do not contain mathematical concepts. For instance, in the scaled triangles problem, Aleix asks ‘puc posar un terç de a [parameter which represents the length of a segment] aquí?’ He is not sure about the syntax of the tool circle given the centre and the radius. He explores the case in which the radius is defined as a fraction that contains a parameter (1/3a).

As these students are autonomous and they have their own control-skills, the tutor gives non-requested messages of level one, only if the students are not aware of a false statement or there is evidence of a lack of understanding. For instance, in the root problem, Aleix generalizes the similarity criteria AAA of triangles and applies the criterion AAAA for quadrilaterals which is false. The tutor gives first a validation message of level one to make the student react. Aleix is not aware of the fact that the analogy is not correct; he states again ‘Si els angles són iguals la forma ha de ser la mateixa’. The tutor gives then a cognitive message of level two to prompt Aleix to search for a counterexample. Aleix reacts to this message and modifies the criterion. He obtains a similarity criterion for quadrilaterals. The fragment shows a shift in the structural competence as consequence of the task, tutor and student interactions.

Finally, as part of the didactic performance, the tutor gives messages of level one to elicit students’ thinking without suggesting mathematical contents. For instance, some of these messages are ‘Estàs segur?’, ‘A què et refereixes?’, ‘Com ho podries justificar?’ Some of these messages also have the purpose of obtaining feedback from the students’ reasoning processes. For instance, in the quadrilateral problem Aleix states ‘Llavors...he de veure
que les àrees són iguals.‘ As there is no feedback about this statement, the tutor asks Aleix about the statement of the conjecture as we can see in the paragraph below.

Tutor: Com has arribat a la conjectura de que les àrees són iguals?
Aleix: He vist que l’àrea d’això [PMN] no depèn de P. Llavors aquesta àrea és constant i com aquesta és constant, aquesta també [(CPN) + (PMA)]. La part de sota és independent de la part de dalt (PAM). [Figures 5.13, 5.14 and 5.15]
Aleix: També he considerat un cas particular: P el punt mig. Com que no importa on estigui P puc considerer aquest cas. He vist que totes les altures són iguals per paral·leles llavors he suposat que les àrees són iguals […]

Aleix also states that he had the idea of considering constant areas through the following analogy; As the figure AMNC reminds him of the case of a right angled triangle inserted in a circumference whose diameter is the hypotenuse, he is persuaded to think in terms of invariants and transforms the circumference into a line, as we can see in the figure below (Figure 5.9). The invariant in the case of a circumference is the right angle and the preserved areas inside the half-circumference. The invariant in the quadrilateral problem is the area. Aleix frequently uses analogies to state conjectures.

Another example of this tutor’s messages occurs in the scaled problems triangle. Aleix has made a mistake, as he tries to obtain the trisection of the median considering the circle of centre A and radius $a/3$. He is misled by the labels ($a$ does not represent the length of the median) and does not consider that he should define the radius $2/3a$ to obtain the point P such that $3PM=AM$ (visual difficulty in transferring simple number theoretic properties such as addition and multiplication to quantities seen as segments). Despite the fact that Aleix knows that the construction does not lead to the expected
construction, the tutor gives him a validation message to foster an evaluation phase and to obtain feedback.

5.1.3 Results on the synergy of environments

In this section we formulate the results on the synergy of environments for each prototypic case (not confident, confident and autonomous profiles). We structure our analysis of the synergy results in terms of two dimensions. The first dimension is related to the evolution of our four geometrical competences in the course of the actual learning trajectory. We systematically compare the actual evolution of every competence between the (only) paper-and-pencil environment and the technological environment (i.e., paper-and-pencil and GGB). We expect in this way to show the concurrence of paper-and-pencil and GGB environments at work.

The second dimension points to the dynamics of the interaction between paper-and-pencil and GGB in every micro-cycle in which the use of the software is allowed. We show that the different phases of the resolution process of a problem in a technological environment are significatively different from the phases identified only in a paper-and-pencil environment.

During the teaching experiment, when students face a problem in the technological environment, the tutor requests them to construct with GGB the figure associated to the statement of the problem. In the previous GGB introductory session, these students have learned that every figure they could construct should be validated by the so-called dragging test. This is a strong constraint, since as we shall see, the students have necessarily to work out in depth the familiarisation phase of the resolution process for a given problem. This prompts the students to confront conflicts because they know that the figure associated to the statement has to be robust (in terms of the dragging test). This is not an obvious task for the students. These conflicts represent a very valuable source of learning, namely an insight into the structural competence (logical structure of the statement of a problem), as well as the visual competence (figure versus drawing). What we have said basically represents the set of commonalities among the different profiles.

5.1.3.1 Results on the synergy of environments: ‘not confident students’

As mentioned before, in the familiarization phase they use the software to construct the figures associated to the statement of the problem. They try to construct robust diagrams and are aware of the scheme ‘dragging to validate a construction’. Nevertheless, they have difficulties in discerning which elements have to be dragged. These learning conflicts have as a consequence that the students spend more time structuring the problem. In this time, the average student of this profile switches to the strict paper-and-pencil when the exploration phase starts. After writing in the worksheet a summary of the observed data of the problem, they immediately leave the worksheet and continue the exploration phase with GGB. The exploration phase with the help of GGB is characterised by a rather naïf discrete dragging. They try to obtain particular examples and reason on the static figure with the help of (GGB) measure tools. At the end of this
exploration, they switch again to the worksheet and only use sporadically GGB for very simple validations of properties based on measures tools or perceptual approach.

This behaviour is also observable in paper-and-pencil resolutions. For instance, in the quadrilateral problem, Marta does not visualize dynamic variations of the point P along the base of the triangle. She reasons on concrete cases by drawing different triangles and by considering also particular cases.

In the construction problems, they use dragging combined with measure tools for adjusting’. This corresponds to the heuristic strategy of considering the problem solved, and they try to find geometric properties that lead them to the construction of a proof. This strategy turns out to be observed only when the students are allowed to use GGB.

As we already said, these students focus on measure tools (distances, areas, angles, etc) combined with the tool dragging. For instance, in the median problem, Marta conjectures through perceptual apprehension that both heights are equal, but she validates it with measure tools. They trust that measures obtained with the software are exact and do not distinguish the GGB plane from the Euclidean plane. They have difficulties interpreting the round-off error. In the scaled triangles problem, Marta has difficulties conjecturing the property of the centroid, namely that it is a point of trisection of the median, because she does not drag the vertices of the triangle and only obtains measures from a concrete triangle. She does not interpret the round-off error and thus she is not aware of the relation between the segments defined on the median by the centroid.

The students also use the software during the execution phase for validating the results obtained or to visualize the ratios between segments. For instance, in the first question of the scaled triangles problems, Marta tries to apply Thales theorem in two-steps to prove the equality of segments. During the execution phase, she drags the point P along the median to discern equivalent ratios on the screen.

The instrumentation of the scheme ‘dragging to find invariants’ helps these students to understand the logical structure of the problems. Nevertheless, they only drag the objects in a small area. For instance, in the scaled triangles problem, Marta understands by dragging the point P along the median that the equality of segments (EM = MF) holds for any point P of the median. She also understands the dependence of the points E and F observing the indirect dragging.

We summarize in the following paragraph the most relevant results concerning the three aspects analysed in the previous sections: a) the hypothetical learning trajectory, b) the orchestration and c) the synergy of environments.

There are no relevant changes in these student’s cognitive students concerning the procedures used to solve the problems that may be due to a-priori conceptions of the students. They try to obtain equality of ratios to apply area formulas, but they also tend to reason on figures. We identify the transition to a higher degree of visual, structural and instrumental competences. The tutor’s orchestration has a key role in this shift, as it encourages students to use the drag tool progressively to understand the distinction of drawing and figure and to acquire the instrumented scheme ‘dragging to validate a property’ and ‘dragging to find a counterexample’. The instrumentation of these schemes helps the students to state and validate conjectures and to understand the structure of the problem. The tutor’s messages are mainly messages of levels two and three. These students are not autonomous and require frequently cognitive and contextual messages.
They abandon their resolution strategies easily. When working in a technological environment, they show a strong confidence in the measurements shown by the software. They validate their conjectures with measure tools combined with dragging. They do not feel confident using the software and use only the basic tools required for the constructions. We recognize situations described in literature as false negative, which is a situation in which “empirical data contradict a true statement” (Chazan, 1993, p.364). This arises from the accuracy and precision limitations of measurement devices. The tutor has a key role in helping these students to interpret the round-off errors. They use dragging for adjusting in construction tasks when they have difficulties to construct a robust figure, but they have difficulties in interpreting the round-off error. The fact that they avoid certain robust constructions may also be due to a lack of confidence in exploring new tools.

5.1.3.2 Results on the synergy of environment: ‘confident’ students
These students start the resolution of a problem as the previous profile. They also use the software mainly in the familiarization, analysis/exploration and validation phases. They try to make robust constructions and they have instrumented the scheme ‘dragging to validate a construction’. Nevertheless, they drag the objects in a small area. They also use the grid and the coordinate axes to construct the figures. In consequence, they always fix one of the vertices to the origin of coordinates and the other vertex to the x-axis. They tend to base their conjectures on a perceptual approach. They have instrumented the scheme ‘dragging to find invariants’. They use continuous dragging (Olivero, 2002), but they do not drag the initial triangle. This kind of dragging is significatively more sophisticated than the one used by non-confident students (discrete dragging), and is used mainly in the exploration phase to search for key-lemmas. At this moment of the resolution process, they shift to the worksheet and reason on the static figure. They do not leave the paper-and-pencil environment, unless they have difficulties in their current resolution strategy.

For instance, in the resolution of the median problem, Guillem drags continuously the point P along the median and deduces that the equality of areas is equivalent to the equality of heights (simple reduction lemma). These students use the instrumented schemes ‘cinema dragging to analyse the variations of the figure through motion’ and ‘dragging combined with perceptual approach to validate a conjecture’. They pose and test conjectures with the help of GeoGebra, but they avoid the use of measure tools. This fact may be due to their reluctance to construct new elements with GeoGebra that may overcrowd the figure. For instance, in the resolution of the median problem, Guillem is reluctant to construct the feet of the heights. Also, in the scaled triangles problem, he avoids the construction of auxiliary parallel lines, as we can observe in the following paragraph.

[Guillem drags the point P to obtain visually the trisection of the segment]  
Guillem: ara s’ha de justificar  
Guillem constructs the other median and observes that the point P belongs to the median  
Guillem: I si fes una altra paral·lela, aquests tres segur que també serien iguals. Però ja hi han masses coses en el dibuix…
If Guillem had constructed the parallel lines, he would have observed the trisection of the other sides of the triangle that leads to the construction of a proof without using the key-lemma (properties of the centroid).

The tutor’s orchestration has a key role in the instrumental genesis of the software. The students have difficulties in discerning free/dependent GGB objects. The understanding of this distinction helps them to understand the motion dependency between the elements of the problem and also the logical structure of the problem. For instance, in the resolution of the scaled triangles problems, Guillem has difficulties with these notions, as we can see in the paragraph below.

Guillem: No ho puc dibuixar perquè no tinc les dades, P varia. Hauria de fer les paral·leles que es puguin moure.
Guillem: Hauria de moure P…. [He remains in silence observing the figure on the screen]
[...] [He tries to construct the parallel lines before constructing the point P.]
Tutor: per utilitzar l’eina paral·lela a una recta per un punt has de definir el punt.
Guillem: Però, si poso el punt, quedarà fix
Tutor: No, un punt qualsevol de la mitjana es pot desplaçar sobre la mitjana.
[Guillem constructs with GGB the point P on the median and the parallel lines through P to the sides of the triangle.]
[Guillem drags the point P along the median to trisect the segment visually, observing the grid. Guillem tries to observe on the algebraic window the length of the segments, but the segments EF and FM are not defined]

These students avoid the use of measure tools (distance, angle) and validate their conjectures by dragging combined with a perceptual approach. Nevertheless, they do not distinguish the difference between the GeoGebra plane and the Euclidean plane. For instance, Guillem is aware of the lack of precision of a paper and pencil drawing but uses the following instrumented scheme to validate a property (parallelism).

To check that two lines (PQ) and (RS) are parallel, he constructs a parallel line through the point P to the segment [RS]. As he perceptually observes that the point Q belongs to the line, he deduces that these lines are parallel. This action reveals that he is not aware of the fact the GGB plane has a finite number of points. Instead, when working with paper and pencil he does not trust the property. He also has difficulties understanding Thales reciprocal. At the beginning, these students had difficulties understanding the properties of the software.

We summarize in the following paragraph the most relevant results concerning the three aspects analysed: a) the hypothetical learning trajectory, b) the orchestration and c) the synergy of environments.

There are no relevant changes concerning these student’s cognitive structures, concerning the procedures used to solve the problems. These students tends to use algebraic resolution strategies based on applying theorems to obtain linear elements of the figures and then compare the expressions obtained for the areas. There is a shift in the visualization competence. They gradually start to visualize dynamic variations of the elements through the instrumented scheme ‘dragging to find invariants’. They avoid the use of measure tools and immediate perceptual approach can be an obstacle. They avoid overcrowding GGB constructions. This may be due to a lack of confidence using the software and also to the fact that they do not tend to reason on figures. Despite not using
measure tools, their techniques reveal that they have difficulties understanding the difference between the GGB plane and the Euclidean plane (technique to check parallelism).

The tutor has a key role in encouraging these students to validate their immediate perceptual conjectures before starting to prove these conjectures deductively. There is also a shift in the structural and instrumental competence. The problems proposed in a more generic way, the tutor’s orchestration and the synergy of environments all foster this shift in the geometrical competences. These students react to messages of level one and the tutor has a key role. The students progressively react to the tutor’s messages and start to consider generic figures and to drag some elements of the figure.

5.1.3.3 Results on the synergy of environments: ‘autonomous’ students

These students use the software throughout the resolution process; they are confident using the software and try to explore new tools and the different windows (algebraic and geometric window). This fact helps them to connect different resolution approaches. They develop a wide range of instrumented schemes and use the software in all the phases of the resolution process. In opposition to what has been observed in the previous profiles, the autonomous students combine constantly in a parallel way the worksheet and GGB. Even if an autonomous student has solved or reached his (current) objective in the worksheet, he tries other resolution approaches in both environments. The verification phase turns out to be very rich.

They understand the difference between the GeoGebra plane and the Euclidean plane and avoid the use of measure tools for validating conjectures. They develop other instrumented schemes to validate geometric properties. For instance, Aleix does not measure angles to validate perpendicularity of two lines or equality of angles. He uses the GGB tool that gives the relation between objects, and if there are round-off errors due to the construction process he drags the elements in a wide area to explore the round-off behaviour. If he considers that the round-off error can be ignored, he then tries to prove deductively the property. There is a constant shift between empirical exploration and deductive reasoning.

These students make a simultaneous use of the geometric window and the algebraic window to connect different concepts. They drag the elements of the geometric window trying to find algebraic-geometric invariants. For instance, in the resolution of the scaled triangles, Aleix trisects the median constructing the circle of centre A and radius \( \frac{d}{3} \) (one third of the length of the median). He drags the vertices of the triangle in a wide area while observing the changes in the circle’s equation to validate the construction. The software also allows them to explore different representations of mathematical objects.

These students understand the characteristics of the geometric objects constructed. As stated by Olive, Makar, Hoyos, Kor, Kosheleva and Strässer. (2008), “a common feature of dynamic geometry software is that geometric figures can be constructed by connecting their components; thus a triangle can be constructed by connecting three line segments. This triangle however, is not a single, static instance of a triangle that would be the result of drawing three line segments on paper; it is in essence a prototype for all possible triangles. By grasping a vertex of this triangle and moving it with the mouse, the length
and orientation of the two sides of the triangles meeting at the vertex will change continuously “(Olive et al., 2008, p.12).

These students understand the GeoGebra objects. For instance, Aleix changes the orientation of the triangles through dragging. Instead, Guillem (‘confident’ profile) uses the tool polygon to construct a triangle and avoids using segments as he fears that the figure may be ‘messed-up’ when dragging its vertices. Moreover, he uses only continuous linked dragging of a point along a segment.

The students characterized in the third profile are autonomous students. They use the software in the sense considered by Olive et al. (2008), “as a laboratory science, mathematics becomes an investigation of interesting phenomena, and the role of mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanation for the phenomena”(Olive et al., 2008, p.13).

We summarize in the following paragraph the most relevant results concerning the three aspects analysed: a) the hypothetical learning trajectory, b) the orchestration and c) the synergy of environments.

There are changes in these student’s cognitive structures, concerning the procedures used to solve the problems. They connect different resolution approaches and the tasks, the tutor’s orchestration and synergy of environments all have a key role. There is a shift in the four geometric competences. These students are autonomous and do not request the help of the tutor. The tutor has a key role encouraging these students with messages of level one to find other resolution approaches and to explore the software. They try to explore new tools and connect conceptual structures as a result of the interaction between the three elements (tasks, orchestration, and synergy of environments). They understand the limitations of measure tools and distinguish situations such as false negative and false positive (Chazan, 1993). They internalize the dragging as a theoretical control. We also observe different instrumented schemes such as: ‘dragging to explore the round-off error behaviour’ with the intention of discerning false negative and false positive situations, ‘algebraic-visual invariant guessing’, which is based on the simultaneous use of the algebraic window and the geometric window.

In the following section we compare and discuss the results for each profile considering the four geometrical competences and the students’ behaviours.

5.2 Discussion

In this section we compare results concerning the behaviours of the three prototypic profiles and the acquisition of geometrical competences trying to find differences and similarities between these results that may provide us with insight into the two research questions. We report and reflect first on differences and then we reflect on commonalities.

Relevant differences among the different profiles

There are differences between the three profiles (not confident, confident, and autonomous) concerning the indicators studied (geometric competences and behaviours). We do not observe changes in the students’ cognitive structures regarding the procedures
used to solve the problems for the not confident and confident students. The not confident students tend to base their resolution strategies on applying theorems to obtain equality of ratios and then apply the area formulas. They also base their reasoning on figures. Nevertheless, they only consider equicomplementary dissection rules for congruent figures and the tutor has a key role to foster this resolution approach. The confident students tend to use symbolic representation for segments and base their resolution strategies on algebraic approaches based on comparing the algebraic expressions of the areas obtained. Instead, for the autonomous students there are changes in their cognitive structures. They connect different resolution approaches. For example, they connect strategies based on applying transformations that preserve area or dilatations and the properties of the area function. They also connect strategies based on obtaining equality of ratios. The tasks, the synergy of environments and the tutor’s orchestration all have a key role. For instance, these students react to messages of level one, which encourage them to explore the software and the task.

For the not confident and confident profiles, the synergy of environments, the tutor’s orchestration and the tasks do not lead them to consider strategies based on equicomplementary dissection rules (for similar and equivalent triangles). There is no evidence of a shift in the deductive competence for these two profiles. Instead, for the autonomous students there is evidence of a shift concerning the deductive competence.

These students prefer to reason ‘on figures’ and try to avoid the use of algebraic resolutions. They use the drag tool as a search mode, trying to find invariants that lead them to the construction of a proof. They explore the features and constraints of the software and develop instrumented schemes to carry out the tasks. “Hölz (2001) distinguished two ways of using the mediating functions of the drag mode: the test mode and the search mode. All his observations led him to conclude that the second use of the drag mode is not a short term affair but results from a learning process that is characterized by different layers of conceptions” (Laborde, Kynigos, Hollebrands and Strässer, 2006, p.287). Autonomous students are at a deeper layer of conceptions than not confident and confident students. The two former profiles have difficulties in using the drag mode as a search mode, and they progressively start to drag elements but in a small area, and they seldom drag the outside vertices of the triangles (quantifier with respect to the outside triangle). They use the search mode only for bounding dragging. This fact obstructs the consideration of other resolution approaches, such as resolution approaches based on equicomplementary dissection rules for equivalent figures. Instead, autonomous students drag all the ‘draggable’ elements of the figure in a wide area through cinema-dragging (for example, they change the orientation of the figures).

The synergy of environments, the tutor’s orchestration and the tasks help them to find deductive justifications and to connect concepts. “The use of dragging allows one to experience motion dependency that can be interpreted in terms of logical dependency within the DGE, but also interpreted in terms of logical dependency within the geometrical context (i.e. logical dependency between geometrical relationships within a geometry theory). The difficulty of establishing a correct and effective interpretation of different kind of movements is well documented (Hölz, Hoyles & Noss, 1994; Jones 2000; Talmon & Yerushalmy 2004), nevertheless such a ‘correct’ interpretation constitutes the
basic element for effective use of dragging tool for both conjecturing and proving, i.e. for producing conjectures and their mathematical proofs” (Mariotti, p.196, 2006). Moreover autonomous students distinguish between “behaviour that results as a consequence of the tool design and behaviour that is a direct result of mathematics” (Laborde, Kynigos, Hollebrands and Strässer, 2006). The observed instrumented schemes prove this understanding. For instance, we have identified the instrumented scheme ‘dragging for exploring the round-off behaviour’ to distinguish false positive and false negative situations (Chazan, 1993).

Instead, not confident and confident students have difficulties in making this distinction. During the teaching experiment, they start to make this distinction as a result of the interactions tasks-tutor-environment. Scher (2001) found that students in fact did not make those distinctions. “Ways to address this dilemma include the careful design of tasks and the milieu that is not restricted to technology and a focus on the critical role of the teacher” (Laborde, Kynigos, Hollebrands & Strässer, 2006). During the teaching experiment, the tutor has a key role in fostering this distinction for the not confident and confident profiles. The tasks and the synergy of environments also have a key role.

As another relevant difference among the three profiles, we distinguish the different uses of the measure tools. The way of using measure tools has consequences in the acquisition degree of the visual, instrumental and structural competences.

Not confident students use systematically measure tools and tend to trust measure tools more than their reasoning. The tutor has a key role in helping them to construct the relation between spatial-graphical level and the theoretical level, and in understanding the dependency relationships in GeoGebra. They start to use measurements in deductive arguments.

Confident students use a systematically perceptual approach combined with bounding dragging to state and validate conjectures. They avoid the use of measures because they would have to make longer constructions to obtain the required measures. For instance, to obtain the altitude’s length of the triangle, they have to construct the feet of the height, and this requires longer constructions that overcrowd the GGB figure. These students are confident about their immediate perceptual approach and try to avoid over-crowding the figure as it obstructs the visualization of geometric properties. This can be an obstacle, as they have not internalized the dragging tool as a theoretical control. They also have difficulties in discerning the GGB plane from the Euclidean plane. The tutor has a key role in helping these students to overcome these difficulties.

Autonomous students combine the use of measure tools observed in the algebraic window, the tool ‘compare objects’ and the perceptual approach. They are aware of the difference between the GGB plane and the Euclidean plane and develop instrumented schemes such as ‘dragging to explore round-off error’. We also observe the instrumented scheme ‘algebraic-visual invariant guessing’ based on the simultaneous use of the geometric window, the algebraic window and the dragging tool in a wide area.

Finally, concerning the HLT, we conclude that:
For not confident and confident students, we should propose more problems to encourage the use of equicomplementary dissection rules for non congruent figures and to foster the instrumental genesis of the dragging tool. This requires a longer teaching experiment.
We should also set higher level problems for autonomous students to allow them to reach a deeper understanding of the concepts and higher acquisition degree of geometrical competences. Concerning the tutor’s orchestration there are also relevant differences concerning the three profiles.

Not confident students: the messages of the tutor have a key role, but they are mainly messages of level two and level three. This is due to the fact that these students tend to react to messages of level one by requesting new messages. In order to promote students’ autonomy, the messages should be modified and the tutor should give more messages of level one, or he should propose problems of lower level such as key lemmas.

Confident students: they react to messages of level one and do not tend to request messages.

Autonomous students: the tutor’s orchestration is adequate, as it fosters exploration and insight into the geometric competences. These students tend to avoid requesting messages as they want to solve the problems on their own. Validation messages are relevant as they encourage the students to explore the software and the tasks, and thus provide opportunities for learning.

We have identified several differences among the three profiles concerning the acquisition of geometric competences and the student’s behaviours characterized by the hypothetical learning trajectory, the orchestration and the synergy of environments. Nevertheless, there are also commonalities that give us insight into the research questions. In the following section we consider these commonalities.

**Relevant commonalities among the different profiles**

Concerning the geometric competences, there is a shift in the visual, structural and instrumental competences for all the students, and we identify learning opportunities for the three profiles.

Not confident and confident students overcome difficulties in visualizing the exterior height of a triangle, and they start to think in variations of the point and the figure quantified universally. This fact helps these students to pose and test conjectures and to understand the distinction between figure and drawing. Nevertheless, not confident students tend to reason on static figures. We observe this behaviour in both environments. They tend to use discrete dragging to create particular cases.

The use of GeoGebra, the tutor’s orchestration and the tasks also have a key role in the understanding of the logical structure of the problems for the three profiles.

Confident students also overcome difficulties in visualizing the exterior height of a triangle and progressively start to think about variations of the universally quantified elements. At the beginning of the teaching experiment, these students had difficulties understanding the need for considering a generic figure, whereas in the fourth problem (quadrilateral problem) they consider a generic triangle and visualize dynamic variations of the point P along the side of the outside triangle.

Autonomous students start to apply reconfigurative operative apprehension (transformation of two shapes in one equivalent shape) and to visualize algebraic-geometric relationships of segments as quantities. They connect different resolution approaches and they acquire insight into homothetic transformations. They understand the relation between the area function concept and the similarity congruence, and equivalence of figures. The tasks, the synergy of environments and the tutor’s
orchestration have a key role. They also get a deeper understanding of the logical structure of the problems.

The role of dragging is relevant for posing and validating conjectures for all the students’ profiles, and in the case of not confident students we observe the transfer to the paper-and-pencil resolutions. They start to consider particular cases as a heuristic strategy for posing and validating conjectures. Instead, in the root problem, these students had difficulties in the resolution process due to the fact that they did not try to state a conjecture. Moreover, they did not understand the dynamic structure of the root problem. We also identify common strategies such as investigating the problem through constructions done by eye or based on dragging to adjusting. “Healy (2000), comments how the two kinds of construction are complementary [soft/robust]: The general emerges in the exploration of soft constructions (…). The cycle 'soft then robust' seems to be a driving force behind students’ generalization processes” (Laborde, Kynigos, Hollebrands and Strässer, 2006, p. 289).

Concerning the tasks proposed, we consider that some modifications are necessary. Firstly, the insertion of the figure in a grid obstructs student’s understanding of the logical structure of the problem. We conclude that we should modify the statement of these problems by presenting the figures without a grid.

- Root problem: we maintain the concrete lengths of the outside rectangle because we use this problem as an initial test. The use of a generic rectangle may foster strategies based on equicomplementary dissection rules. As we do not give the concrete position of the point E on the diagonal, we do not foster strategies based on algebraic resolutions or analytical geometry as it also requires the understanding of the logical structure of the problem (parameters). We should add other questions for students who do not have difficulties in solving the problem, such as autonomous students. For instance, we may ask about the relation between the areas of the outside rectangle and the inside rectangles. This requires knowledge about similar quadrilateral and the relation between the ratio of areas and the squared factor of similarity.

- Scaled triangles problem: this problem has a key role in fostering the understanding of the logical structure of the subsequent problems. The fact that it is of a constructive nature (existential quantifier) encourages the students to use the drag tool. Nevertheless, all the students have difficulties in visualizing the algebraic-geometric relations of segments as quantities. We should include more problems to foster the algebraic-geometric visualization. Neither the grid nor the first question helps the students to overcome these difficulties. Nevertheless, this task has a key role in the instructional design. We may add a question concerning areas of both triangles to foster the understanding of the relation between similarity and area.

- Median problem: we should modify the labels of the triangle’s vertices. As GeoGebra assigns labels for the points in alphabetic order the students construct the oblique median. We may ask the students to consider different medians by
changing the orientation of the triangle through dragging. It may help them to visualize strategies based on equicomplementary dissection rules and would encourage students to use the drag tool in a wide area.

- Quadrilateral problem: All the students have difficulties in proving Thales reciprocal. This is due to the fact that these students are not used to this strategy of proving. We should include other problems to introduce proof by contradiction. For instance, Leung and López-Real (2002) stressed the key role of dragging in forming a mathematical conjecture and even in proving by contradiction. The students do not relate the areas of the inside figures with the area of the outside triangle. This is again a visual-algebraic difficulty. We may add a second question concerning the relation between the inside quadrilateral and the outside rectangle. We should also design more problems to help the students to overcome this difficulty. It would require a longer teaching experiment. Also, it would be interesting to generalize the problem considering other ratios (instead of midpoints theorem).

In the following table we summarize the commonalities and differences described in this section (Table 5.13).
<table>
<thead>
<tr>
<th>Profiles</th>
<th>Competences</th>
<th>Students’ Behaviours</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Visual</td>
<td>Structural</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>NOT CONFIDENT</strong></td>
<td>They progressively consider particular cases for the point and the triangle (two quantifiers) (+)</td>
<td>They progressively understand the logical structure of the problem (++)</td>
</tr>
<tr>
<td></td>
<td>Visual-algebraic difficulties: segments as quantities (-)</td>
<td>Similar resolution strategies: They do apply strategies based on equicomplementary dissection rules if the figures are congruent (o)</td>
</tr>
<tr>
<td><strong>CONFIDENT</strong></td>
<td>They start to visualize dynamic variation (linked variation) (+)</td>
<td>Understanding of the logical structure of the problem (++)</td>
</tr>
<tr>
<td></td>
<td>Visual-algebraic difficulties: segments as quantities (-)</td>
<td>Learning opportunities: height of a triangle, equicomplementary dissection rules for congruent figures (+)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Figural inferences</td>
</tr>
</tbody>
</table>

**Figural inferences**

There is no evidence of a shift in this competence. The insertion of the figure in a grid in the problems obstruct the understanding of the logical structure of the problems. Difficulties with proof by contradiction.
<table>
<thead>
<tr>
<th>AUTONOMOUS</th>
<th>Reconfigurative operative apprehension (++)</th>
<th>Deep understanding of the logical structure of the problem (++)</th>
<th>Spontaneous conjecturing and self-initiated effort to validate the conjectures (++)</th>
<th>There is a shift in the competence. More elaborated deductive proofs. Partial insight into the proof by contradiction, Proof by equivalence (+)</th>
<th>They do not pay attention to superfluous numerical data of the problems. We should propose generalizations of these problems or higher level problems as they are able to reach a deeper understanding of the mathematical concepts involved and a higher acquisition degree of geometric competences</th>
<th>Mainly messages of level one that fosters the exploration. They seldom request messages and they are autonomous. The tutor has a key role to encourage them to explore the software and the tasks, and thus to provide opportunities for learning. They use GGB throughout the resolution process and use it for the construction of a proof: (instrumented arguments) Instrumentation of the drag tool. Search for algebraic-geometric invariants Instrumented schemes ‘dragging to explore the round-off error’, ‘algebraic-visual invariant guessing’</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Partial algebraic-geometric visualization: segments as quantities (+)</td>
<td>They connect different resolution strategies</td>
<td>Learning opportunities: similarity criteria for quadrilateral (+)</td>
<td>Internalization of dragging as theoretical control (+)</td>
<td>They do not pay attention to superfluous numerical data of the problems. We should propose generalizations of these problems or higher level problems as they are able to reach a deeper understanding of the mathematical concepts involved and a higher acquisition degree of geometric competences</td>
<td>Mainly messages of level one that fosters the exploration. They seldom request messages and they are autonomous. The tutor has a key role to encourage them to explore the software and the tasks, and thus to provide opportunities for learning. They use GGB throughout the resolution process and use it for the construction of a proof: (instrumented arguments) Instrumentation of the drag tool. Search for algebraic-geometric invariants Instrumented schemes ‘dragging to explore the round-off error’, ‘algebraic-visual invariant guessing’</td>
</tr>
<tr>
<td></td>
<td>Similar resolution strategies: They do apply strategies based on equicomplementary dissection rules if the figures are congruent (o)</td>
<td>(+)</td>
<td>equivalent and similar figures</td>
<td>competence a consequence of these interactions (tutor-environment-tasks)</td>
<td>elements but only continuous bound dragging or dragging in a small area Difficulties to differentiate GGB plane and Euclidean plane</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.13: Differences and similarities between the three profiles
6. Conclusions and discussion
We structure this chapter in three sections, which are as follows: results, discussion, weaknesses of the study and didactical implications. We first summarize the findings in relation to the research questions and the main goals of the research. Secondly, we address our main conclusions about our work. We then report some of the possible weaknesses of the study. Finally, we consider the didactic implications.

6.1 Results concerning the main goals
Guided by the initial research questions, we now present the results of our work. For the sake of clarity, we formulate again our research questions:

- How can the use of GeoGebra be integrated into a teaching sequence to promote students geometric competences (visual, structural, instrumental and deductive)?
- What are the students’ behaviours (term specified in section 1.3 Research questions) when solving problems under the influence of the instructional activities, the teacher’s orchestration and the synergy of paper-and-pencil and GeoGebra?

Concerning these research questions and the aims of the research, we specify the results obtained in the following paragraphs.

Results concerning the first objective: Design of an instructional sequence of problems based on different itineraries of problems
The results concerning the first goal included the elaboration of an instructional design especially adapted to an itinerary based series of problems which were well-suited for a technological environment. It required a careful definition of the concept of root problem and itineraries, the selection of similar problems that compared areas and distances of plane figures, and the a priori analysis of these problems. We defined the basic space of each problem (Cobo, 1998), its pedagogical space and its level of difficulty, and finally we set forth in the context of problem solving the meaning of what we call logical structure of a problem. In this way we were able to define the similarity relation between problems. We stated then the similarity relation between problems in terms of the basic space, the logical structure and the level of complexity. This careful and detailed construction of the instructional design should then be considered as a relevant contribution of our research.

Results concerning the second objective: Gaining insight into the acquisition of geometric competences in the context of an hypothetical learning trajectory
With respect to the second objective, we essentially identify two findings.
Firstly, following Drijvers (2003), we created a coding system (a discrete scoring, namely four scores) with indicators for each score and competence in order to increase our insight into mathematical competences. These indicators allowed the researchers to report and qualify transitions or changes in the four aforementioned competences of students. It should be pointed out that the indicators for the scores were especially defined for the characteristics of the problems of the instructional design; in this case, plane geometry
problems worked out by the students either with paper-and-pencil or in a technological environment.

Secondly, the feed-forward of the local cycle\textsuperscript{13} (actual learning trajectory) confirmed that the problems selected for this research were in fact well designed, since we reported the expected learning transitions that could arise in the problems. In particular, we could analyse the evolution of students in their acquisition of mathematical competences; namely the visual, structural and instrumental competences, whereas the deductive dimension did not experiment any significative improvement. This last point could well be due to the temporal shortness of the teaching experiment. As the learning trajectories have an iterative character, the actual learning trajectories observed suggest several modifications of the problems that could be a priori fruitful for posterior hypothetical learning trajectories: a) to avoid problem statements with inserted figures in a grid, b) to generalise the problems statements to encourage more visual resolutions strategies, c) to consider problems with more complex logical structure (for instance more nested quantifiers) that could promote the understanding of practical mastering of the dichotomy free (and partially free or linked) objects versus dependent objects.

\textit{Results concerning the third objective: Analyzing and characterizing the learning trajectories in terms of the transitions, tutor’s orchestration and synergy of environments.}

We characterised three prototypic profiles of students in terms of the learning trajectories according to the transitions between the micro-cycles, the tutor’s orchestration and the synergy of environments. We now summarise the characteristic of the three profiles: confident, not confident, and autonomous students.

\textit{Not confident students:}
Concerning the visualization competence, these students progressively started to think about variations of the elements universally quantified. They progressively considered particular cases. At first they had difficulties in distinguishing properties of the \textit{drawing} from properties of the \textit{figure} and did not think about variations of the initial figure. We identified some of the difficulties discussed by Presmeg (1986a; 1986b) which involved: (1) the one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even lead to false data. (2) An image of a standard figure may induce inflexible thinking, which prevents the recognition of a concept in a non standard diagram. The use of GeoGebra, the problems proposed (dynamic problems) and the tutor’s orchestration helped these students to progressively consider \textit{dynamic imagery} (Presmeg, 1986a) and to overcome partially the previous difficulties. They progressively considered “discrete variations of the elements” in both environments (paper-and-pencil and technological environment) to produce different examples. They used these examples to state and check conjectures. When working with GeoGebra, these students tried to construct \textit{robust constructions} and validated their conjectures (equality of areas, equality of lengths, parallelism, etc.) through \textit{photo-dragging} combined with measure tools. They stopped dragging after a short time and concentrated on the interpretation of the static figures. For the paper-and pencil problems, they tried to validate their conjectures

\textsuperscript{13} The local cycle trajectory actually corresponds to a single itinerary.
through perceptual approach by drawing some examples. These students were probably aware of the necessity of proving, but they had difficulties in elaborating conceptual justifications.

With respect the deductive competence, we have no evidence of improvement. They used relationships observed in the examples to justify their conjectures. We mainly identified crucial experiments. We distinguished several types of crucial experiments, identified by Marrades and Gutiérrez (2000) such as: example-based, constructive, analytical and intellectual justifications.

During the teaching experiment there were few changes in their cognitive structures concerning the procedures for solving the problems. This could be due to the difficulty in discerning similar and equivalent triangles. The use of GeoGebra did not encourage these students to explore other resolution strategies. They mainly used measure tools to state and validate conjectures, but they had difficulties in differentiating the GeoGebra plane from the Euclidean plane (difficulties to interpret the round-off error). These students were not confident about their ability to solve problems and trusted GeoGebra measurements more than their reasoning. They frequently requested validation messages to the tutor. Moreover, they reacted to messages of level one by requesting new messages. The tutor mainly gave messages of level two and level three. The tutor appeared to be essential in the learning process, but it would have been better to propose messages of lower level in order to promote students’ autonomy.

**Confident students**

In this profile, there was also an improvement in the visual, structural and instrumental competences. These students progressively started to consider variations of the elements and used them in their resolution strategies. We identified dynamic imagery (Presmeg, 1986a). They visualised continuous variations of the elements and used it as part of their resolution process. The dynamic visualization helped them to state and to validate conjectures, but it did not promote resolution approaches based on equicomplementary dissection rules. They still focused on algebraic approaches. We should set more problems to encourage strategies based on discerning equivalent and similar figures. Despite the apparent preference of algebraic resolutions over visual resolution strategies, the students performed their proof search with the so-called figural inferences. The use of GeoGebra had a key role in the elaboration of figural inferences. There was no evidence of an improvement of their deductive competence. They did pragmatic justifications.

The problems set, the tutor’s orchestration and the synergy of environments helped these students to improve their visual, structural and instrumental competence. These students progressively started to drag elements and developed instrumented schemes based on the use of the dragging tool (cinema-dragging) combined with perceptual approach.

**Autonomous students**

There was an improvement in the visual, structural, instrumental and deductive competence. Concerning the visual competence, we identified dynamic imagery (Presmeg, 1986a). They visualized continuous variations of the elements and considered reconfigurative dynamic visualization. We define this term as the visualization processes
that transform several shapes in a single shape that has the same area. During the teaching experiment, they connected different resolutions strategies. These students internalized the drag tool as theoretical control. They developed a wide range of instrumented schemes and used GeoGebra throughout the resolution process. They used the drag mode as a test mode and a search mode to find geometric invariants. Concerning the deductive competence, these students elaborated deductive justifications. We also identified transformative justifications and structural justifications (Marrades & Gutiérrez, 2000). During the teaching experiment they developed awareness of different formal strategies of proving, such as proof by contradiction.

We called this type of students autonomous because they did not tend to request messages (relatively few interactions with the tutor\textsuperscript{14}), and they explored successfully the problems and the software on their own.

We have found evidence that the use of GeoGebra, in the context of the proposed tasks, helped the students in the process of visualization. We have no evidence of relevant improvement in the deductive competence for not confident and confident students. We hypothesize that a longer teaching experiment would be necessary.

Concerning the visualization competence, the students did not acquire the ability to visualize algebraic relations of segments as quantities. The use of GeoGebra did not support this ability. We conjecture that the hypothetical learning trajectory might treat it separately. We should set specific problems to help the students to gain insight into this aspect.

6.2 Discussion

We now discuss the results obtained and report our conclusions concerning the research. As already mentioned, there is a close match between the students’ hypothetical and actual learning trajectories concerning mathematical competence. This close match is not coincidental. As Stylianides and Stylianides (2009) conclude in their research about the use of HLT to facilitate the transition from empirical arguments to proof, the close math between the HLT and the ALT “is an outcome of the systematic approach within the design-based paradigm to develop the focal and other instructional sequences” (p.347). In their research, they focus on the instrumental competence. In our case, we have tried to be more fine-grained in the concept of mathematical competence, because we have considered four competences; namely visual, instrumental, structural and deductive. We extend their work on the assumption that the instructional design is implemented with the help of a technological environment and the role of a tutor who has orchestrated the teaching experiment. We report for each competence the relation between the HLT and the ALT, where the use of our coding system based on indicators (related to each competence) turns out to be crucial.

\textsuperscript{14} In fact these students tended to avoid the tutor’s suggestions and preferred to tackle the problem as a personal challenge.
Geometric competences

Visual competence
The visualization competence is singular with respect the other competences, since we claim that visual competence can be seen at work in the structural, instrumental and deductive competences. The ID was designed in such a way that probably there would be situations in which the visual competence would be the source of learning conflicts. A paradigmatic example is the difference between figure and drawing. The figure that a student visualizes is in fact a mental image of the properties that apply to a given problem. The ID in its ALT has shown repeatedly that such (learning conflict) situations have indeed occurred. Moreover, the use of GeoGebra has amplified the visual competence in action, as students have continuously faced the problems that the ontological status of geometrical objects (dependent versus free object) has in GeoGebra. GeoGebra is a visual tool which offers and challenges the student with the multiple information it gives. Setting aside the free/dependent distinction, in some phases of the teaching experiment students have shown a feeling of uncertainty about the possibility of dragging certain objects, as we know that not every object in GeoGebra is a priori draggable. In the search for a solving strategy (or proof), the visual competence may again play an important role. The dynamic character of the selected problems of the ID and the use of GeoGebra has repeatedly shown the visual competence at work. Finally, visualization may appear as a genuine proof. One simply has to consider the so-called proofs without words.

In this teaching experiment no visual proof has been observed. This however must not be seen as a drawback of the teaching experiment. On the contrary, proofs without words are not easy. Very clever and original reasoning is found behind them. One wonders whether some of the figural inferences and visual analogies observed in the teaching experiment are close to some visual proof reasoning. We may claim (see cognitive characterization we survey below) that the ID jointly with the use of GeoGebra and the help of the tutor have provided an opportunity of real learning for the students.

Structural competence
The instructional design has been carefully elaborated in order to display the structural competence at work. The problems are non-routine problems. Their logical structure is mainly composed of at most two nested universal quantifiers. These problems are useful for highlighting the difficulties and misunderstandings of the students. They are confronted with situations that provoke changes in their conceptions. The use of GeoGebra turns out to be helpful as a visual and algebraic tool for helping the students to understand the structure of the problem. GeoGebra has also helped the students to see the process of structuration of the problems from another point of view. We especially have in mind the interesting analogy of dependent/free object with objects quantified universally and the terms they bind.

The tutor’s orchestration has also contributed to the improvement of the structural competence. The tutor implements scaffolding strategies and initiates situations for institutionalization (Brousseau, 1981), that is, “a situation that aims to point out and give an official status to pieces of knowledge that were constructed during a class activity and that can be used in a future work (see Balacheff, 1990, p.260)”.

Instrumental competence
Concerning the instrumental competence, the students see the limitations of empirical arguments as methods for validating mathematical generalizations. As we see it, this is due to the fact that the structural competence and the instrumental competence are adjacent in the cognitive plane. We show that the students understand the necessity of proving as a consequence of the obstacles encountered to elaborate the problems. One of the students stated ‘com no diu res puc agafar un triangle rectangle’ (potential cognitive source of conflict in the understanding of the universal quantifier). One of the students, who had a strong mathematical background, developed complex instrumented schemes that allowed him to explore the parameters of the problem. He used what we define as massive continuous dragging of all possible elements in a wide area to explore the parameters of the problem. He searched for algebraic-geometric invariants by trying to detect true properties, but he was aware of the need for proving.

Deductive competence
From the overall competences, we think that the deductive competence turns out to be the weakest one in the sense that we have observed less evidence of actual learning. Nevertheless, this does not mean that there has been no improvement at all in deductive dimension. An interesting example is the improvement in the internalisation of the role of the universal quantifier. In the analysis we have reported signs of enhancement of the (ontological) understanding of generic examples, which we know is at the root of a formal proof involving a universal quantifier. In the ALT we have also observed improvement of deductive skills in one student, namely Aleix’s understanding of the concept of proof by contradiction. An interesting HLT for further research would be to focus on the competences related to the deductive dimension, i.e. our deductive competence (defined in the theoretical framework) and its strongly related instrumental competence. As we see it, an ID which focuses on the deductive competence should take into account the instrumental competence. As an example, we show that some students are able to manipulate algebraic expressions, but they are not aware of the meaning of equivalences. Through a chain of equivalences Guillem obtained a tautology, but he had difficulties in interpreting it going backwards through the sequence of equivalences and arriving at the truth of a statement.

We have focused on practical issues, such as evidence of actual improvement of mathematical competences. Our aim now would be to determine the correct range of application, or contribution of the theoretical framework we have assumed. We show some benefits of our theoretical assumptions. We have shown how to design an instructional sequence, which contributes significatively to the improvement of mathematical competences. Our theoretical assumptions were to select a small self-contained set of problems (in our case) of plane geometry, which had to satisfy several constraints: a) a carefully chosen topic of the thematic approach (in our case a deep interrelation between similarity theory and Euclidean area), b) a series of similar problems, where by “similar” we understand a sameness in terms of logical form, a controlled degree of complexity and finally a similar basic space. As we planned to carry out a research in an orchestrated domain with the help of a technological environment, we had to add more theoretical constraints to our instructional design. On the one hand, a
controlled (small) set of messages intended to (minimally) help the students in the learning process. On the other hand, we as researchers selected problems that would be well-suited for the concurrence of software of dynamic geometry (GeoGebra). In sum, we believe that one benefit of our theoretical framework is how to design an instructional sequence which has a pedagogical value in the context of mathematical learning. It is then a means of completing the HLT in such a way that it “provides a framework and thus a theoretical elaboration of the HLT construct” (Simon & Tzur, 2004, p.91). This in turn shows that the use of the HLT concept is an adequate research tool for analyzing the development of our hypothesis and expectations.

Finally, concerning the problem of the learning process, another contribution of our research has been to give a fine-grained set of mathematical competences analysed with a codification system, adapted from Drijvers (2003), which has given us evidence of mathematical learning.

**Synergy of environments**

As Kieran and Drijvers (2006) found in their study, technique and theory emerge in such a way that very complex interactions may appear in the resolution of problems. In our research, we focus then on the concurrence of paper-and-pencil and GeoGebra techniques.

We give an account of the synergy of the technological and the paper-and-pencil environment which naturally arises in the four competences. The most obvious evidence of synergy of both environments may be found in the visual competence. GeoGebra is a visual tool which constantly displays algebraic and geometric knowledge. When a student faces a problem with the help of GeoGebra, he is naturally constrained to take into account some mathematical knowledge that a priori without the tool would not emerge. GeoGebra names and structures mathematical objects without the request of the student. We expect then that a possible co-emergence of both environments when the student exploits the feedback he gets from GeoGebra. Some examples would be situations in which the students have used simultaneously information from the algebraic and geometrical windows in the phase of structuration. A very important feature of software of dynamic geometry is the dragging of objects. We have observed several kinds of dragging from discrete case (to obtain particular cases or counterexamples and reason on static figures) to the continuous case (to search for algebraic-geometric invariants). A source of interesting co-emergence of technique and theory is to be found in the ontological distinction which GeoGebra makes of dependent/free objects. In this way, the student is confronted with an information intimately related to the logical structure of the problem which otherwise could have not considered. An example would be the situation in which the students have requested from the tutor whether a given object is draggable. As we know, only free objects (and partially free or linked objects) are draggable, and this is strongly related to the understanding of the logical structure.

The instrumental competence can benefit from the synergy of environments. A paradigmatic example would be the case of Aleix, which we can consider as an example of massive continuous dragging. The proof search crucially is influenced by the GeoGebra environment, for we observe how the student (Aleix) tries to identify invariants in the algebraic and geometric window, which may represent the source of interesting conjectures such as key-lemmas (see competence indicators). We believe that
Aleix’s behaviour constitutes a not obvious use of the technological environment in the process of mathematical learning. We may conclude that the synergy of environments, the tasks proposed and the tutor’s orchestration help the students to improve the visual, structural and instrumental competence. In the following section we point out some weakness of the research.

6.3 Weaknesses of the study and further research

We consider in this section the weaknesses and limitations of this study and point to further research expectations. There are limitations concerning the design phase, the teaching experiment and analysis phase. Also, the results of this study can not be generalized, as we carried out the teaching experiment in a particular high-school and we follow a qualitative analysis with a reduced sample. The results can not be extrapolated to other situations.

- We have considered in the instructional sequence the definition of different itineraries of problems, taking into account the difficulty levels defined in the second chapter. Nevertheless, it is not possible to extrapolate this way of assigning levels to other problems. The level of difficulty of a problem depends on many factors that can not be considered a priori.
- The students are novice users of GeoGebra and the hypothetical learning trajectory is designed for a period of four lessons. We should consider a longitudinal teaching experiment. As stated by Artigue (2002), “the construction of the instrument, instrumental genesis, is a complex process, needing time, and linked to the tool characteristics and to the subject’s activity, his knowledge and former method of working.”
- During the teaching experiment the students worked individually. We took this decision to analyse the individual use of GeoGebra. Also as we planned to analyse the interactions student-task-tutor-environment, the consideration of another agent made in depth analysis of the learning trajectories difficult. Nevertheless, we would have obtained more feedback if the students had worked in pairs. Also we consider that students can benefit from their interactions by working in pairs.
- The tutor’s orchestration is difficult to implement in a whole-group class as the tutor can not observe the resolution processes of all the students. Also the tutor’s instrumentation process may be considered as it influences student’s instrumentation process.
- The prototypic behaviours observed in this local cycle (third itinerary of problems) can vary for other problems and other contexts.

As further research, we plan to consider a full research cycle, taking into account the results of this first research cycle. We plan to design a longer teaching experiment for a better understanding of the synergy of environments and the effect of the instructional design on the students. Taking into account the feed-forward from this first teaching cycle (design-experiment-analysis), we plan to modify the instructional design to start a new cycle. We also plan to carry out this second teaching cycle with more students to obtain a characterization of other learning trajectories.
6.4 Didactic implications

The design of instructional sequences allows the integration of GeoGebra into the teaching and learning of geometry. As stated by Laborde et al. (2006), it is not only the interaction of the students and the software that matters, but also the design of tasks and the learning environment. The concept of hypothetical learning can help teachers and researchers to design successful teaching sequences which integrate the use of dynamic geometry software. These tasks should be designed to foster the construction of instrumented schemes. For instance, the problems proposed should focus on developing questions about motion.

As regards orchestration, we conclude that contextual messages should be based on messages of level one and two. A most productive form of learning takes place after the instrumented techniques have provided some kind of confrontation or conflict with the student’s expectations. Kieran and Drijvers (2006) found, in the context of CAS use, that the students’ seeking for consistency evoked theoretical thinking and further experimentation. They also concluded that the teacher is essential in this process. “Without the teacher orchestrating the theoretical and technical development of the task situation, and asking key questions at the right moment, the advances of the students would likely have been less dramatic” (Kieran & Drijvers, 2006, p.255).

The synergy of environments (dynamic geometry software and paper-and-pencil) can have a profound effect on helping students to develop geometric competences. The problems that compare areas and distances of plane figures can help the students to develop awareness of different resolution approaches, and to gain a deeper understanding of the concepts of distance and area, namely to internalize the interrelation between similarity theory (ST) and Euclidean area (EA), which were considered in the second chapter (2.4 Thematic approach). The design of itineraries and the tutor’s orchestration should focus on these aspects.
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