Combination of dynamic geometry, algebra and calculus in the software system GeoGebra

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July 2004

Abstract: Dynamic geometry and computer algebra systems have highly influenced mathematics education. Unfortunately, these tools have been totally unconnected. GeoGebra is a new software system that integrates possibilities of both dynamic geometry and computer algebra in one tool for mathematics education.

ZDM classification: R 20, U 70

1. Introduction

Dynamic geometry systems (DGS, see [18, 21, 22]) like *Cabri* or *Cinderella* and computer algebra systems (CAS, see [6, 9]), such as *Mathematica*, *Maple* or *Derive* have highly influenced mathematics education. The teachers' training curriculum in mathematics at the University of Salzburg has been responsive to these developments. Algebra systems (mainly *Derive*, *TI-92* and *Mathematica*) and the dynamic geometry system *Cabri* have been introduced in special lectures.

On the basis of prototypical examples university students have been presented the benefits and possibilities of the different kinds of software [10]. The variations of parameters in the algebraic representation and the ensuing effects on the graphs have been investigated using computer algebra systems. But more and more it turned out that the abilities of students to generate prototypes of mathematical objects [7] necessitated the converse way too. This means studying the influence on the algebraic representation by dynamical manipulations of the geometric objects.

In 1997 Karl Fuchs gave a lecture on the use of the TI-92 calculator in mathematics education at the University of Salzburg (see [12]). This calculator already offered both DGS and CAS but these two parts were completely separated programs. During this time, Markus Hohenwarter, one of Mr. Fuchs' students, suggested a closer connection between visualisation capabilities of CAS and the dynamic changeability of DGS. This wish for a bidirectional combination of dynamic geometry and computer algebra had already been stated before ([18], [19]).

> The three solution protocols [graphical, numerical and algebraic] should not be considered separate, but rather as constituting a holistic comprehensive computer-aided approach. [19, p. 324]
>
> There is a need for further software development to provide a single package combining the desired features. [19, p. 337]

In 2001, Markus Hohenwarter began the work for his master's thesis *GeoGebra - a Software System for Dynamic Geometry and Algebra in the Plain*. The aim of this project was to develop a completely new kind of tool for mathematics education in secondary schools.

GeoGebra already received several international educational software awards: European Academic Software Award 2002 (Ronneby, Sweden), L@rnie Award 2003 (Vienna, Austria), digita 2004 (Cologne, Germany) and Comenius 2004 (Berlin, Germany).

2. What is GeoGebra?

GeoGebra is an interactive geometry software [13, 14] that also offers algebraic possibilities like entering equations directly. It is aimed at students (aged 10 to 18) and teachers in secondary schools. The program encourages students to approach mathematics in an experimental way. For example, it is possible to investigate the parameters of a circle's equation by dragging the circle with the mouse. On the other hand, students may also manipulate the equation directly and see the changed circle in the geometry window.

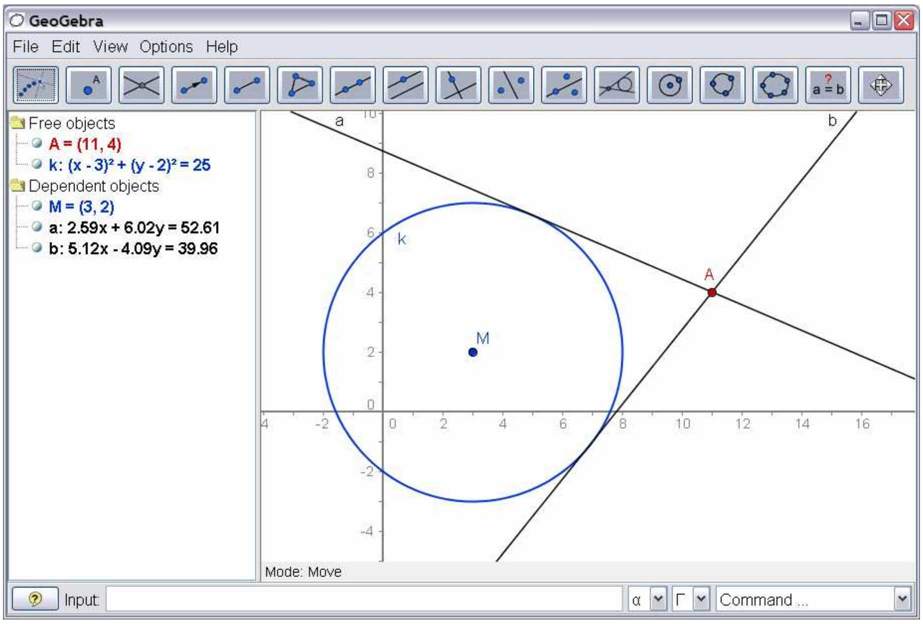

Figure 1. Circle with tangents

The interactive construction protocol is indeed significant. Already Bender and Schreiber pointed out that the description of a geometric construction is the pretheoretical basis for the idea of an algorithm [3]. The construction protocol of *GeoGebra* makes it possible to redo constructions at any time, insert new elements and even change its order with hindsight. Whenever students are entering or deleting expressions they must be aware of functional dependencies [23].

3. What does GeoGebra offer?

The basic objects in *GeoGebra* are points, vectors, segments, polygons, straight lines, all conic sections and functions in *x*. With *GeoGebra* dynamic constructions can be done like in any other dynamic geometry system. These constructions may be altered dynamically by dragging free objects. Furthermore, it is possible to enter coordinates of points or vectors, equations of lines, conic sections or functions and numbers or angles directly.

Hence, from the very beginning the software has been designed for the use in schools. The treatment of problems should not be affected by system-caused translations. Manipulations ought to be possible in a familiar way [5]. Big efforts have been made to allow input in school notation: For example a line g may be entered as $g: 3x + 4y = 7$ or a circle c as $c: (x - 3)^2 + (y + 2)^2 = 25$.

Also, calculations with geometric objects like points and vectors are feasible. The centroid of a triangle with vertices *A*, *B* and *C* might be entered as *S = (A + B + C) / 3*. Additionally, *GeoGebra* offers many powerful commands starting from the slope of a straight line up to differentiation and integration of functions.

GeoGebra is multilingual not only in its menus but also in its commands. For example, the English command *Intersect* becomes *Schneide* in German and *Interseccíon* in Spanish.

4. Applications in Schools

GeoGebra is a very versatile tool for mathematics education in secondary schools. In teaching mathematics it might be used in many different ways.

1. *GeoGebra* for demonstration and visualisation

   Even in traditional teaching, computer software has its status. In his discussion about the role of specific software Becker [2] mentions the aspect of specific software as a tool for demonstration and visualisation. In this sense, *GeoGebra* is a software with a wide coverage due to its different representations.

2. *GeoGebra* - a construction tool

   In 1990 Karl Fuchs [8] pointed out the importance of computer aided drawing / designing systems for teaching constructive geometry at the state of the art. Not substitution of traditional but integration of new methods was intended by him. The idea of 'computer utilisation' became fundamental. *GeoGebra* has all the abilities which are demanded from a suitable drawing / designing software [3].

3. *GeoGebra* and discovering mathematics

   Computers and mathematical software have provoked new basic questions on teaching mathematics. Students can organize knowledge on their own. For example, Artigue and Lagrange [1] are reporting on the positive influence of computer algebra systems on teaching mathematics. This experimental form is added to the traditional form of teacher concentrated education as described in item 1 above. *GeoGebra* can be used as an important tool for this challenge. It can help to create a suitable atmosphere for learning (compare with [17]).

4. *GeoGebra* for preparing teaching materials

   GeoGebra encourages teachers to prepare materials for the teaching process using it as a cooperation-, communication- and representation tool. This follows Kerres' ideas of the educational functions of new media [15].

The software can be used with students aged 10 to 18, beginning with simple constructions up to the integration of functions. No matter if students explore mathematics alone or in groups, the teacher should try to be an advisor in the background who gives support when help is needed. The students' results of their experiments with *GeoGebra* should be the basis for discussions in class. This gives teachers more time to concentrate on fundamental ideas and mathematical reasoning [20].

The following example shows a dynamic worksheet where students can discover the function of the slope of a parabola themselves. They see an arbitrary point *T* on the graph, the tangent *t* through this point and its slope *k*. Now we investigate another point (x_T , k) with the *x*-coordinate of *T* and the corresponding slope *k* as its *y*-coordinate. By dragging the point *T* along the graph of the parabola, a trace of the slope function is generated experimentally.

Now, the students should try to read off the equation of the slope function using the coordinate grid. By plotting this equation with *GeoGebra* they can verify their assumption easily.

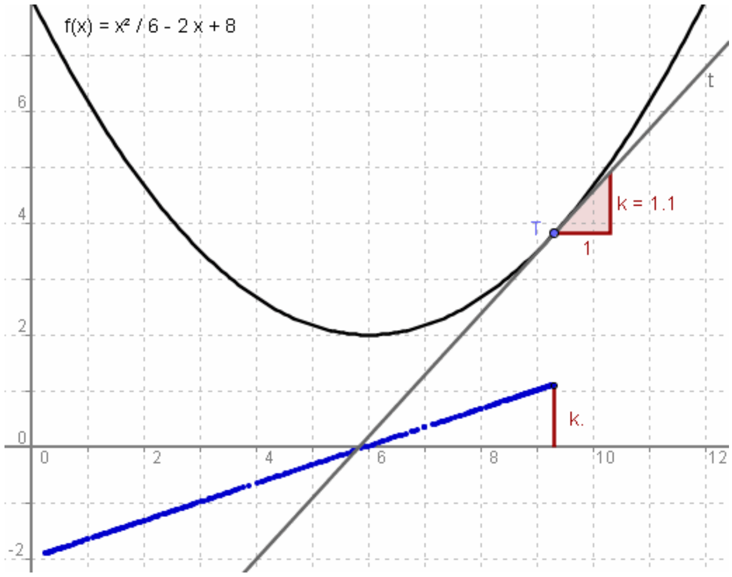

Figure 2. Slope function of $x^2/6 - 2x + 8$

This is a motivating example for the geometric model of the first derivative. Here, the tangent is used as a black box [11, p.57] providing a basis for the discussion of the tangent line problem afterwards.

5. Dynamic Worksheets

With *GeoGebra* it is possible to create interactive HTML pages – so called *dynamic worksheets* – which can be used with any Internet browser that supports *Java* (e.g. *Internet Explorer*, *Mozilla*, *Netscape*). These worksheets are totally independent from the program itself, i.e. *GeoGebra* does not have to be installed to use the worksheet. Thus, *GeoGebra* is also a tool to create interactive e-learning content.

The stand-alone application can be used on any platform (*MS Windows*, *Unix*, *Linux*, *MacOS*) and *GeoGebra* may even be started directly from the Internet eliminating complicated installation or upgrade procedures which is especially useful for computer networks in schools.

6. Behind the Scenes

GeoGebra is based on projective and Euclidian geometry in the real plane. Equations are expanded and simplified symbolically and a special grammar for arithmetical expressions in school notation has been implemented. In most dynamic geometry systems you will encounter jumping objects in certain constructions. For example, the intersection points of two conic sections may permutate wildly in *Cabri* when an ellipse becomes a hyperbola. In *GeoGebra* this continuity problem [16, pp.84] is tackled by a sophisticated heuristic approach [13, pp.163] eliminating jumping objects.

For the differentiation and integration of functions in one variable the open source computer algebra system *JSCL* is built into *GeoGebra*.

7. Future Aspects

The development of *GeoGebra* by Markus Hohenwarter is going on rapidly in the course of his PhD thesis *GeoGebra - Development of Educational Material and Applications in Mathematics Teaching*. Some of the planned features are:

- intersection of functions, function – line
- root finding of arbitrary functions, local minimum / maximum, turning point
- locus lines
- automatic animations
- macros

The free software and further information can be found on the *GeoGebra* website on the Internet *http://www.geogebra.at*.

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